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SOME PROBLEMS INVOLVING THE RAY APPROXIMATION OF THE WAVE EQUATION

BY

ZHENG BAO-SHAN

(C)

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEACH

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OF MASTER OF SCIENCE

IN

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Abstract

In this paper the conditions under which the equation of motion in a 3-D inhomogeneous isotropic medium can be approximated by the scalar wave equation, and the solution to the scalar wave equation approximated by rays are investigated. The relationship between these conditions is also discussed.

Accurately and efficiently solving the eikonal equation which describes the kinematic characteristics of the wave motion is one of the more important problems in ray approximation methods. It is convenient to solve the eikonal equation as a boundary value problem in some geophysical problems. Much attention is given to the investigation of Pereyra's method of solving a boundary value problem, which is a variable order finite difference method on nonuniform meshes. These meshes are selected automatically by the program as the computation proceeds. We have programed this method for 2 and 3 dimensional velocity models with linear gradients. This program is tested for models with linear velocity gradients in one and two space variables. This method yields satisfactory results when compared with known analytic and numerical solutions given by W. H. K. Lee's programs.

A new method based on the numerical solution of the eikonal equation is presented in this thesis.

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1. Introduction

Problems of wave propagation play an important role in various branches of geophysics. But so far, because of the complexity of the Earth's crust and the uppermost mantle, no analytical solutions of the equation of motion are known. There are two main approaches to the investigation of wave fields in such complicated media: approximate methods and methods based on direct numerical solution of the elastodynamic equation (such as the finite difference method).

Approximate methods can be used to study the propagation of elastic waves in quite complicated structures. Let us mention here at least Whitham's wave theory and various types of high-frequency approximations, such as the ray method.

Ray theory has been extensively used in seismology (see Hron, F. 1984). The ray method can be applied to more complicated media, in which other approximate methods cannot be used. The major problem in the application of the ray method is that it represents only a high frequency approximation and as such has a number of serious restrictions. Nevertheless, at the present time it is the only method that is able to give us an approximate answer to many problems of seismic body wave propagation in complicated geological models.

It is, therefore, very important to know the conditions under which the ray method approximation can be used to compute the wave field.

One of objectives of this paper is to discuss the restrictions for using the ray method. This problem will be discussed in sections 2 through 5.

In order to study the inhomogeneous structure of the Earth, seismologists have developed several techniques to trace seismic rays. So far there are two methods for tracing seismic rays between two given end points in three dimensional media. One of these methods involves "shooting" the ray from one point with a given starting direction and then changing this starting direction until the ray emerges at the desired target. It is called the "initial value method". Another method involves "bending" the initial path between the end points until it satisfies the principle of stationary time. This is called the "boundary value method". Some authors, for example, Yang and Lee (1976), Julian and Gubbins (1977), Pereyra, Lee and Keller (1980) have recently presented a comparison of these two methods, concluding that the boundary value method is computationally faster.

As Wesson (1971, p.741) pointed out, tracing seismic rays between two end points is required in seismological applications, such as earthquake location (e.g., Engdahl and Lee, 1976) and the determination of the three-dimensional velocity structure under a seismic array (e.g., Aki and Lee, 1976; Aki et al., 1977).

The second task of this thesis is to investigate the two point seismic ray tracing problem using the boundary value method in sections 6 through 8. We have programed Pereyra's method and have computed results for some models. High accuracy and efficiency are achieved by using variable-order finite difference methods on nonuniform meshes which are selected automatically by the program as the computation proceeds. The variable mesh techinique allows the program to "adapt itself" to the particular problem at hand, thereby producing more detailed computations where they are needed, as in the case of highly curved rays.

The geometrical spreading coefficient which determines the amplitude of the ray may be retrieved easily from the "bending" solution. We will describe a new numerical method to calculate the amplitude of the ray in section 9.

2. The Wave Approximation to the Equation of Motion

The most general form of the equation of motion in a three dimensional inhomogeneous isotropic elastic medium may be written as follows:

$$(1)\quad \rho\partial^2\vec{u}/\partial t^2 = (\lambda+\mu)\nabla\theta + \mu\nabla^2\vec{u} + \theta\nabla\lambda + 2(\nabla\mu\cdot\nabla)\vec{u} + \nabla\mu\times(\nabla\times\vec{u}),$$

where λ and μ are Lames parameters, ρ is the volume density. These quantities are all functions of position. $\vec{u}$ is the displacement vector, and $\theta=\nabla\cdot\vec{u}$ is the volume expansion. Applying the rotation operator ($\nabla\times$) and the divergence operator ($\nabla\cdot$) in turn to (1) yields:

$$(2)\quad \rho\partial^2\vec{\omega}/\partial t^2 + \nabla\rho\times\partial^2\vec{u}/\partial t^2 = \mu\nabla^2\vec{\omega} + \nabla\times[(\lambda+\mu)\nabla\theta] + \\ + \nabla\times(\theta\nabla\lambda) + \nabla\times(\nabla\mu\cdot\nabla\vec{u}) + \nabla\times(\nabla\vec{u}\cdot\nabla\mu),$$

$$(3)\quad \rho\partial^2\theta/\partial t^2 + \nabla\rho\cdot\partial^2\vec{u}/\partial t^2 = (\lambda+2\mu)\nabla^2\theta + \nabla\theta\cdot(\nabla\lambda+\nabla\mu) + \\ + \nabla\mu\cdot\nabla^2\vec{u} + \nabla\lambda\cdot\nabla\theta + \theta\nabla^2\lambda + \nabla\cdot(\nabla\mu\cdot\nabla\vec{u}) + \nabla\cdot(\nabla\vec{u}\cdot\nabla\mu),$$

where $\vec{\omega}=\nabla\times\vec{u}$ is the rotation of the displacement vector. In general, it is difficult to solve Eqs.(2) and (3) directly, although in many cases we are able to simplify them under some conditions and obtain an approximate solution of (1). Now we will consider these approximating conditions. We will study Eq.(3) using the similarity theorem of fluid dynamics (see Tritton, D. J. 1977). It is convenient to rewrite (3) as follows:

$$(4)\quad \rho\partial^2\theta/\partial t^2 + \nabla\rho\cdot\partial^2\vec{u}/\partial t^2 = (\lambda+2\mu)\nabla^2\theta + \nabla\theta\cdot\nabla(\lambda+\mu) +$$

$$+ \nabla\mu\cdot\nabla^2\vec{u} + \nabla\lambda\cdot\nabla\theta + \theta\nabla^2\lambda + (\nabla\nabla\mu):\nabla\vec{u} +$$

$$+ \nabla\mu\cdot\nabla(\nabla\cdot\vec{u}) + \nabla^2\vec{u}\cdot\nabla\mu + \nabla\vec{u}:\nabla\mu\nabla,$$

where ":" indicates the dot product of dyadics. Let us introduce the characteristic dimensions for our problem as follows:

L ..... characteristic length,

T ..... characteristic time (period),

S ..... characteristic quantity of displacement,

$\Theta=S/L$ ..... characteristic quantity of volume expansion θ,

$U=L/T$ ..... characteristic quantity of speed, $V = [(\lambda+2\mu)/\rho]^{1/2}$.

Then we have the following non-dimensional quantities of time t, position $\vec{X}$, speed V, volume expansion θ, and displacement vector $\vec{u}$:

(5) $t' = t/T,$

(6) $\vec{X}' = \vec{X}/L,$

(7) $V' = V/U,$

(8) $\theta' = \theta/\Theta = \theta L/S,$

(9) $\vec{u}' = \vec{u}/S.$

Substituting Eq.(5)-(9) into Eq.(4) and using ∇' as the expression for the Laplacian relative to the coordinates $\vec{X}'$ and making some algebraic operations we have:

(10) $V'^{-2}\,\partial^2\theta'/\partial t'^2 + V'^{-2}(\nabla'\rho/\rho)\cdot(\partial^2\vec{u}'/\partial t'^2) = \nabla'^2\theta' +$

$$+ (\lambda+2\mu)^{-1}\nabla'(\lambda+\mu)\cdot\nabla'\theta' + (\lambda+2\mu)^{-1}\nabla'\mu\cdot\nabla'^2\vec{u}' +$$

$$+ (\lambda+2\mu)^{-1}\nabla'\lambda\cdot\nabla'\theta' + \theta'\nabla'^2\lambda/(\lambda+2\mu) + (\lambda+2\mu)^{-1}\nabla'\nabla'\mu:\nabla'\vec{u}' +$$

$$+ (\lambda+2\mu)^{-1}\nabla'\mu\cdot\nabla'\theta' + \nabla'^2\vec{u}'\cdot\nabla'\mu/(\lambda+2\mu) +$$

$$+ \nabla'\vec{u}':\nabla'\mu\nabla'/(\lambda+2\mu).$$

Now let us discuss (10) instead of (3). Because $\nabla'^2\lambda$ is of the same order as $\nabla'\lambda$; $\nabla'\nabla'\mu$ and $\nabla'\mu\nabla'$ as $\nabla'\mu$, by choosing a suitable S we can get $\partial^2\theta'/\partial t'^2$ to have the same order as $\partial^2\vec{u}'/\partial t'^2$, $\nabla'\theta'$, θ', $\nabla'^2\theta'$, $\nabla'^2\vec{u}'$, $\nabla'\vec{u}'$, and V'. Therefore if we have:

$$\begin{aligned} &|\nabla'\rho/\rho| \ll 1, \\ &|\nabla'\lambda/(\lambda+2\mu)| \ll 1, \\ &|\nabla'\mu/(\lambda+2\mu)| \ll 1, \end{aligned} \tag{11}$$

then all coefficients on both sides of equation (10) except the first term on both sides are << 1. So neglecting all terms in (10) which are << 1, we obtain:

$$V'^{-2}\,\partial^2\theta'/\partial t'^2 = \nabla'^2\theta'. \tag{12}$$

Returning (12), which is written in a non-dimensional form, into the dimensional one we get:

$$V^{-2}\,\partial^2\theta/\partial t^2 = \nabla^2\theta, \tag{13}$$

where $V=[(\lambda+2\mu)/\rho]^{1/2}$ is a function of position.

Note that since

$$\nabla'[(\lambda+2\mu)/\rho] = \nabla'(\lambda+2\mu)/\rho - (\lambda+2\mu)\nabla'\rho/\rho^2, \tag{14}$$

we have:

(15) $\nabla' V^2/V^2 = 2\nabla' V/V = \nabla'(\lambda+2\mu)/(\lambda+2\mu) - \nabla'\rho/\rho.$

Using conditions (11) we have:

(16) $|\nabla' V/V| << 1.$

Returning (16) into the dimensional form and taking the wave length L as the characteristic length we have:

(17) $|L\ \nabla V/V| << 1.$

From the above discussion we know that condition (16) is not always equal to condition (11), but if we have conditions (11), we may certainly deduce condition (16), and also deduce condition (17).

As a result the equation of motion (1) can be simplified to the following scalar wave equation:

(18) $V^{-2}\partial^2\phi/\partial t^2 = \nabla^2\phi,$

assuming the validity of conditions (11).

3. The Ray Equation Approximation of the Wave Equation

In C. B. Officer's book (1974) the conditions under which the wave equation can be approximated by a ray equation are discussed in detail. Let us present his results here.

Suppose that the solution of Eq.(18) has the simple form:

$$(19) \quad \phi = \Phi(x_j)e^{i(\omega t - k_o\Psi(x_j))}, j= 1,2,3$$

where, $k_o=\omega/V_o$, V_o is a reference velocity, and ω is the circular frequency. The necessary relation between Φ and Ψ are obtained after substituting Eq.(19) into Eq.(18) and equating real and imaginary parts:

$$(20) \quad (\nabla\Psi)^2 - (k/k_o)^2 - \nabla^2\Phi/(k_o^2\Phi) = 0,$$

$$(21) \quad 1/2\ \nabla^2\Psi + \nabla\Phi\cdot\nabla\Psi/\Phi = 0,$$

where, $k=\omega/V$.

If the following condition is valid

$$(22) \quad |L^2\ V''/V| << 1,$$

Eq.(20) becomes:

$$(23) \quad (\nabla\Psi)^2 = n^2.$$

That is the so-called eikonal equation, where $n=V_o/V$. In Eq.(22) we have:

(24) $V'' \simeq \partial^2 V/\partial x_1^2$.

The eikonal equation is a first order differential equation. Its solution

(25) $\Psi(x_1,x_2,x_3) = \text{constant}$,

represents a surface in a three dimensional space. By choosing the travel time, τ, as a parameter Eq.(25) can be rewritten for different times as

(26) $\Psi(x_1,x_2,x_3) = \tau$.

The normals to this wavefront define the direction of wave propagation in any isotropic medium. These are called the rays or ray paths. The ray solution, then is a complete solution to any particular propagation problem within the validity of the approximation of the eikonal equation to the wave equation.

The equation for the normals to the wavefront; i.e. the ray path equation, which will be produced in detail in section 6 is:

(27) $d(n\, d\vec{r}(x_1)/ds)/ds = \nabla n$,

where $\vec{r}(x_1)$ is a position vector of the point (x_1) on the ray. The differential ds is an incremental length along the ray path.

We will call (22) the ray approximation condition, as only under this condition we can use the solution to the ray

equation (27) for the description of the kinematic properties of the wave equation (18).

4. The Energy (Amplitude) Equation

We know from the discussion in section (3) that Eq.(20) and Eq.(21) are equivalent to the scalar wave equation (18). From Eq.(20) we obtained the eikonal equation (27) which describes the kinematic characteristics of motion. Here we shall discuss the energy characteristics of motion using Eq.(21).

Choosing the arc length s of ray and q_2 and q_3 as the ray parameters a general curvilinear coordinate system is obtained (see Fig.1). Here q_2 and q_3 are the Gauss coordinates of the curved surface. The unit vectors tangent to the coordinate lines at the point along the ray are:

$$\begin{aligned} \vec{e}_1 &= \partial\vec{r}/\partial s, \\ \vec{e}_i &= \partial\vec{r}/\partial q_i, \end{aligned} \qquad (i = 2, 3) \tag{28}$$

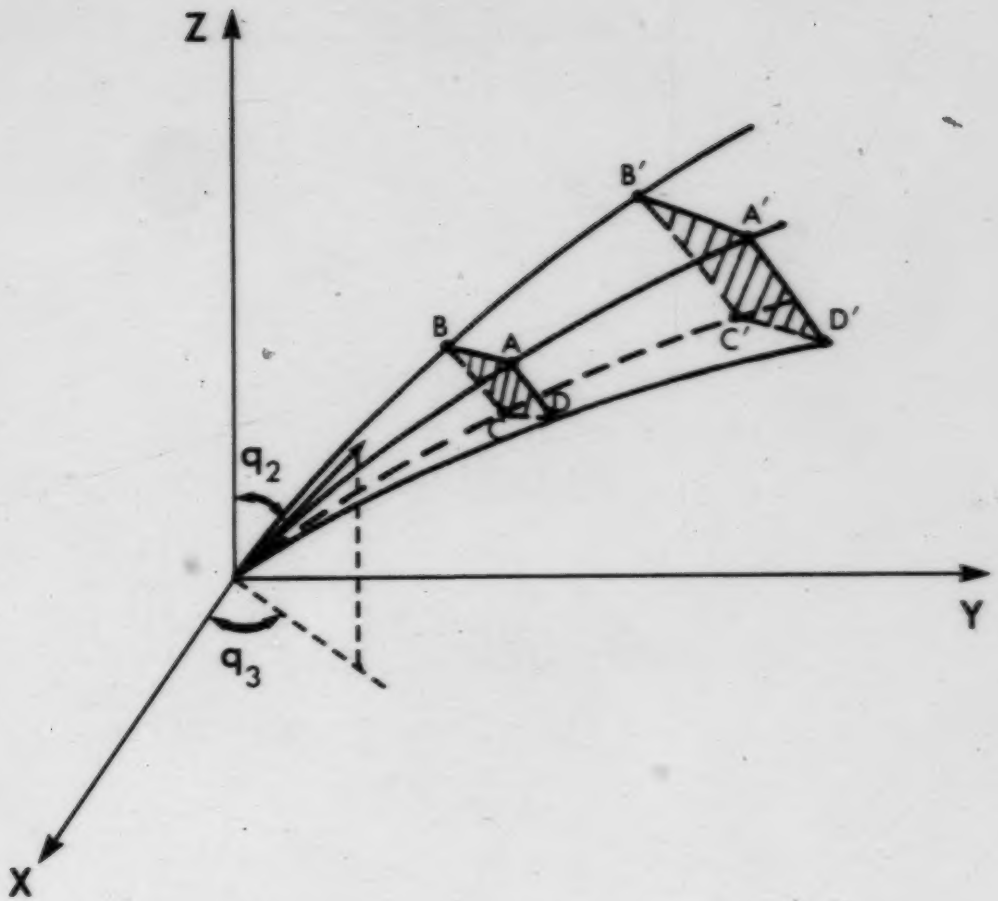
where $\vec{r}$ is the position vector of the point on the ray under consideration.

Lame's coefficients in our curvilinear coodinate system are:

$$\begin{aligned} h_1 &= 1, \\ h_2 &= [(\partial x_1/\partial q_2)^2 + (\partial x_2/\partial q_2)^2 + (\partial x_3/\partial q_2)^2]^{1/2}, \\ h_3 &= [(\partial x_1/\partial q_3)^2 + (\partial x_2/\partial q_3)^2 + (\partial x_3/\partial q_3)^2]^{1/2}. \end{aligned} \tag{29}$$

In the above coordinate system we have:

FIGURE 1. Curvilinear coordinate system on the ray.


Z
B'
A'
B
A
C'
D'
D
C
q_2
Y
q_3
X

(30) $\nabla\Psi = (\partial\Psi/\partial s,\ 0,\ 0)$,

(31) $\nabla^2\Psi = (h_2h_3)^{-1}[\partial(h_2h_3\partial\Psi/\partial s)/\partial s]$,

(32) $\nabla\Phi = (\partial\Phi/\partial s,\ h_2^{-1}\partial\Phi/\partial q_2,\ h_3^{-1}\partial\Phi/\partial q_3)$.

As the original scalar wave equation (18) implies the validity of condition (11), we have:

(33) $|(\lambda+2\mu)^{-1}\partial(\lambda+2\mu)/\partial s| \ll 1$.

Taking $V_o=1$ we have $n=1/V$. By using Eq.(23), (30), (31), (32) and (33) we can rewrite Eq.(21) as:

(34) $1/2\ \partial\ln[(\lambda+2\mu)h_2h_3/V]/\partial s + \partial\ln\Phi/\partial s = 0$.

Obviously, the solution of (34) is:

(35) $\Phi = [(\lambda+2\mu)h_2h_3/V]^{-1/2}C_1$,

where C_1 is a constant which is associated with the initial conditions.

In Fig.1 we depicted a small area ABCD at point A, and a small area A'B'C'D' at point A'. Denoting the average energy flux due to a simple harmonic wave of frequency ω by E we have:

$$E = 1/2\ \omega^2[\rho\Phi^2Vh_2h_3dq_2dq_3]_A,$$

where A indicates that every quantity has to be taken at point A. So when we put Eq.(35) into the above formula we have

(36) $E = C_2\omega^2 dq_2 dq_3$,

where $C_2 = 2^{-1}C_1^2$. In the same way we can get

(37) $E' = C_2\omega^2 dq_2 dq_3$,

which leads to

(38) $E = E'$.

This means that the energy is propagating along the ray tube without any leakage through the sides of the tube. In many practical problems we are frequently asked to find the amplitude of the ray (or equivalently the geometrical spreading) along the ray. We usually choose some parameters having practical geometrical meanings as the parameters q_2 and q_3. The two takeoff angles α and β serve this purpose quite well. In a generally inhomogeneous medium the two unit vectors $\vec{e}_2$ and $\vec{e}_3$ are tangent to the coordinate lines perpendicular to the ray, but are not usually perpendicular to each other. Nevertheless, the average intensity of the ray at points A and A' are:

$$I = E/(|\vec{e}_2\times\vec{e}_3|dq_2dq_3)_A,$$

$$I' = E/(|\vec{e}_2\times\vec{e}_3|dq_2dq_3)_{A'},$$

respectively.

Thus we have

(39) $I/I' = |\vec{e}_2\times\vec{e}_3|_{A'}/|\vec{e}_2\times\vec{e}_3|_A = J(s)/J(s_o)$.

where

$$J(s_0) = |\vec{e}_2 \times \vec{e}_3|_A,$$

$$J(s) = |\vec{e}_2 \times \vec{e}_3|_{A'},$$

Here, s stands for the length of the ray path from some reference point along the ray. Moreover, according to the discussion of average energy in unit time about the simple harmonic wave we may express the wave intensities at A and A' as:

$$I = 1/2\ \omega^2 \rho(s_0) \Phi^2(s_0) V(s_0),$$

$$I' = 1/2\ \omega^2 \rho(s) \Phi^2(s) V(s),$$

This leads to the well known relation:

$$\Phi(s) = [J(s_0)\rho(s_0)V(s_0)/(J(s)\rho(s)V(s))]^{1/2}\Phi(s_0). \tag{40}$$

which expresses the ray amplitude as a function of the position s along the ray.

Based on the knowledge of differential geometry we have

$$J = (EG - F^2)^{1/2},$$

where:

$$E = \sum_{i=1}^{3} (\partial X_i/\partial q_2)^2,$$

$$G = \sum_{i=1}^{3} (\partial X_i/\partial q_3)^2, \tag{41}$$

$$F = \sum_{i=1}^{3} (\partial^2 X_i/(\partial q_2 \partial q_3).$$

In practical problems R_1, the radiation function of the source and R_2, the receiver function on the earth's surface have to be considered. Thus we can rewrite Eq.(40) as:

$$(42) \quad \Phi(s) = R_1 R_2 \, [J(s_0)\rho(s_0)V(s_0)/(J(s)\rho(s)V(s)]^{1/2}\Phi(s_0).$$

Note that we have used conditions (11) and (22) during the derivation of Eq.(40) - (42). Hence, the formulae are valid assuming the validity of conditions (11) and (22).

We can see that the energy equation (the amplitude of the ray) requires the same approximating conditions as does the eikonal equation.

5. Discussion on the Relationship of the Approximating Conditions

So far we have used the approximating condition (11), under which the equation of motion (1) leads to the scalar wave equation. We will refer to Eq.(11) as condition (I).

Similar to Eq.(22), under which the scalar wave equation was transformed into the eikonal equation and the amplitude equation, we may write

(43) $\partial V/\partial x_i \simeq V'$.

Then Eq.(17) may be rewritten as follows:

(44) $|L\ V'/V| \ll 1$.

which will be called condition (II).

The above mentioned conditions (22), i.e.

(22) $|L^2\ V''/V| \ll 1$.

will be called condition (III).

In section 2 we discussed the relationship between condition (I) and condition (II). We showed that (II) follows from (I). Now let us discuss the relationship between condition (II) and condition (III).

In other words, let us investigate the relation between the speed of variation of the elastic parameters of the medium (degree of inhomogeneity) and the minimum frequency (maximum wavelength) of a harmonic source whose seismic wave

field can be decomposed into individual ray contributions. According to B. Officer (1974) and the previous discussion, the inhomogeneity of the medium is always determined by a dimensional length. Officer used L/K as the dimensional length and took K=1, where K is a positive constant number and L is one wave length. Let us also take L as our dimensional length.

Therefore, V' and V" placed in condition (II) and condition (III) may be the first order and the second difference instead of the first and the second derivatives, respectively.

(45) $|\delta V/V| \ll 1,$

(46) $|\delta(\delta V)/V| \ll 1,$

where δV and $\delta(\delta V)$ is the difference of V and δV in one wave length, respectively.

Obviously if condition (45) is valid, condition (46) is also valid. So we have the following relationship:

(47) Condition (I) $\Rightarrow$ Condition (II) $\Rightarrow$ Condition (III).

Relation (47) means that if condition (I) is satisfied we can transform the equation of motion (1) into the scalar wave equation (18). Because at the same time, the ray approximation condition (III) is also valid for any weakly inhomogeneous medium and we can use the eikonal equation (23) and the amplitude equation (42) to study the kinematic and dynamic characteristics of the equation of motion,

assuming the validity of condition (I) without any additional restrictions.

6. The Two Point Seismic Ray Tracing Problem in an Inhomogeneous Medium

We will consider here the case of an inhomogeneous isotropic medium. Let us assume that there is a velocity of propagation, $V(x,y,z)$, at each point $P(x,y,z)$ of the inhomogeneous isotropic medium. We further assume that $V(x,y,z)$ possesses continuous first partial derivatives. The following shows the derivation of the ray equation (27), based on Fermat's principle, in detail.

6.1 The Derivation of the Ray Path Equation

Let us draw an arbitrary curve which connects points $A(x_0,y_0,z_0)$ and $B(x_1,y_1,z_1)$. A particle moving from A to B along the path Γ has the assigned velocity $V(x,y,z)$ at each (x,y,z) position, and moves from A to B in time T given by

$$T = \int_A^B ds/V(x,y,z), \tag{48}$$

where ds is an element along the ray path Γ.

According to Fermat's principle the time T taken by the seismic ray to travel from A to B has a stationary value.

Let us introduce a new parameter ξ to characterize the ray path and write

$$\begin{aligned} x &= x(\xi), \\ y &= y(\xi), \\ z &= z(\xi). \end{aligned} \tag{49}$$

An element of the ray path ds is then written as:

$$(50) \quad ds = (dx^2+dy^2+dz^2)^{1/2} = (\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}\, d\xi,$$

where:

$$\dot{x} = dx/d\xi, \quad \dot{y} = dy/d\xi, \quad \dot{z} = dz/d\xi.$$

Eq.(48) becomes

$$(51) \quad T = \int_{\xi_1}^{\xi_2} (\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}/V(x,y,z)\, d\xi,$$

where $\xi=\xi_1$ at A, and $\xi=\xi_2$ at B. This equation is rewritten as follows:

$$(52a) \quad T = \int_{\xi_1}^{\xi_2} W\, d\xi,$$

$$(52b) \quad W \equiv [\dot{x}(\xi)^2+\dot{y}(\xi)^2+\dot{z}(\xi)^2]^{1/2}/V(x(\xi),y(\xi),z(\xi)).$$

Obviously $W(\xi,x(\xi),y(\xi),z(\xi),\dot{x}(\xi),\dot{y}(\xi),\dot{z}(\xi))$ is a function of ξ and has a stationary value. The Euler equation:

$$d(\partial U/\partial Y')/d\xi - \partial U/\partial Y = 0$$

requires

$$(53) \quad \begin{aligned} d(\partial W/\partial \dot{x})/d\xi - \partial W/\partial x &= 0, \\ d(\partial W/\partial \dot{y})/d\xi - \partial W/\partial y &= 0, \\ d(\partial W/\partial \dot{z})/d\xi - \partial W/\partial z &= 0. \end{aligned}$$

Substituting (52b) into (53) we have:

$$d[V^{-1}\dot{x}/(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}]/d\xi - (\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}\partial(1/V)/\partial x = 0,$$

$$(54)\quad d[V^{-1}\dot{y}/(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}]/d\xi - (\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}\partial(1/V)/\partial y = 0,$$

$$d[V^{-1}\dot{z}/(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}]/d\xi - (\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}\partial(1/V)/\partial z = 0.$$

The parameter ξ along the ray path is still arbitrary, so if we set $\xi=s$, the arc length of Γ, then in order to satisfy Eq.(50) we must have:

$$(55)\quad (dx/ds)^2 + (dy/ds)^2 + (dz/ds)^2 = 1,$$

along Γ. Hence Eq.(54) may be simplified to:

$$\begin{aligned} &d(V^{-1}dx/ds)/ds - \partial(1/V)/\partial x = 0,\\ (56)\quad &d(V^{-1}dy/ds)/ds - \partial(1/V)/\partial y = 0,\\ &d(V^{-1}dz/ds)/ds - \partial(1/V)/\partial z = 0. \end{aligned}$$

If we take $\vec{r}(s)$ as the space curve defining the ray path joining the two points A and B, then Eq.(56) amounts to:

$$(57)\quad d(V^{-1}d\vec{r}/ds)/ds = \nabla(1/V).$$

After choosing $V_o=1$ in $n=V_o/V$, Eq.(27) is identical to (57). We shall use the ray equation (57) in the following form:

$$(57')\quad d(u(\vec{r})d\vec{r}/ds)/ds = \nabla(u(\vec{r})),$$

where $\vec{r} = (x,y,z)$ and $u \equiv 1/V(\vec{r})$.

6.2 Two Point Seismic Ray Tracing Equation in 3-D Inhomogeneous Media

Let us first recast equation (57') as:

$$(58) \quad d^2\vec{r}/ds^2 = V(\vec{r})[\nabla u(\vec{r}) - G(\vec{r})d\vec{r}/ds],$$

where,

$$G(\vec{r}) = (\partial u/\partial x)(dx/ds) + (\partial u/\partial y)(dy/ds) + (\partial u/\partial z)(dz/ds).$$

Writing (58) in detail leads to the following formulation for the two point ray tracing problem:

$$(59) \quad \begin{aligned} d^2x/ds^2 &= V(\vec{r})[\partial u/\partial x - G(\vec{r})dx/ds], \\ d^2y/ds^2 &= V(\vec{r})[\partial u/\partial y - G(\vec{r})dy/ds], \\ d^2z/ds^2 &= V(\vec{r})[\partial u/\partial z - G(\vec{r})dz/ds]. \end{aligned}$$

If S is the total length of the ray between A and B, then the boundary conditions are:

$$(60) \quad \begin{aligned} &x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0; \\ &x(S) = x_1,\ y(S) = y_1,\ z(S) = z_1. \end{aligned}$$

The main advantage of using an arc length parameterization lies in the fact that the functions x(s),y(s) and z(s) are single valued, even when rays intersect.

It is convenient to express Eq.(59) as a system of first order nonlinear differential equations. It is known that second or higher order systems of ordinary differential equations can be easily reduced to first order systems by

introducing auxiliary variables. For instance Eq.(59) reduces to:

$$
\begin{aligned}
&dY_1/ds = Y_2, \\
&dY_2/ds = V[\partial u/\partial x - G(\vec{Y})Y_2], \\
&dY_3/ds = Y_4, \qquad\qquad 0 \le s \le S \\
&dY_4/ds = V[\partial u/\partial y - G(\vec{Y})Y_4], \\
&dY_5/ds = Y_6, \\
&dY_6/ds = V[\partial u/\partial z - G(\vec{Y})Y_6],
\end{aligned}
\tag{61}
$$

where,

$$G(\vec{Y}) = u_x Y_2 + u_y Y_4 + u_z Y_6, \tag{62}$$

and the vector $\vec{Y} \equiv (x, dx/ds, y, dy/ds, z, dz/ds)^T$.

Since in many of these calculations the travel time $T=\int u\,ds$ is of interest, a new variable, Y_7, may be introduced to represent the local travel time. This necessitates the introduction of an additional differential equation

$$dY_7/ds = u,$$

with the initial condition

$$Y_7(0) = 0.$$

In this manner the total travel time $T=Y_7(S)$ is computed automatically with the corresponding ray path.

In order to determine the unknown parameter S we change variable s to $\tau=s/S$ and introduce a trivial differential equation $S'=0$, where now the prime (') will denote differentiation with respect to τ. Calling $Y_8=S$, we obtain

the final set of equations:

$$
\begin{aligned}
Y_1' &= Y_8 Y_2, \\
Y_2' &= Y_8 V[\partial u/\partial x - G(Y) Y_2], \\
Y_3' &= Y_8 Y_4, \\
Y_4' &= Y_8 V[\partial u/\partial y - G(Y) Y_4], \qquad 0 \le \tau \le 1 \\
Y_5' &= Y_8 Y_6, \\
Y_6' &= Y_8 V[\partial u/\partial z - G(Y) Y_6], \\
Y_7' &= Y_8 u, \\
Y_8' &= 0.
\end{aligned} \tag{63}
$$

After introducing τ, a particle moving from $A(x_0, y_0, z_0)$ to $B(x_1, y_1, z_1)$ will produce the change of τ from 0 to 1. So the corresponding boundary conditions to Eq.(63) are:

$$
\begin{aligned}
Y_1(0) &= x_0, \\
Y_3(0) &= y_0, \\
Y_5(0) &= z_0, \\
Y_1(1) &= x_1, \\
Y_3(1) &= y_1, \\
Y_5(1) &= z_1, \\
Y_7(0) &= 0.
\end{aligned} \tag{64}
$$

The eighth boundary condition which is needed to completely define the system can be obtained from Eq.(55) which can be expressed in terms of Y as

$$Y_2(0)^2 + Y_4(0)^2 + Y_6(0)^2 = 1, \tag{65}$$

where

(66) $\vec{Y} \equiv (x, dx/ds, y, dy/ds, z, dz/ds, t, S)^T$.

Solving Eq.(63) with the boundary conditions Eq.(64) and (65) will give the solution to the three dimensional, two point ray tracing problem.

6.3 Two Point Seismic Ray Tracing Equation in 2-D Inhomogeneous Media

In the case of a two dimensional inhomogeneous medium we may only consider the two point ray tracing problem in the x-z plane. Eq.(56) becomes:

$$\begin{aligned} &d(u(x,z)\, dx/ds)/ds - \partial u(x,z)/\partial x = 0, \\ &d(u(x,z)\, dz/ds)/ds - \partial u(x,z)/\partial z = 0. \end{aligned} \tag{67}$$

According to Eq.(55) we have:

$$(dx/ds)^2 + (dz/ds)^2 = 1. \tag{68}$$

Let us introduce a new parameter ψ defined by

$$\begin{aligned} dx/ds &= \cos\psi, \\ dz/ds &= \sin\psi. \end{aligned} \tag{69}$$

We find that

$$\begin{aligned} d^2x/ds^2 &= -(dz/ds)(d\psi/ds), \\ d^2z/ds^2 &= (dx/ds)(d\psi/ds), \end{aligned}$$

yielding

$$(70)\quad (d^2x/ds^2)(dz/ds) - (d^2z/ds^2)(dx/ds) = -d\psi/ds.$$

Expanding equations (67) and multiplying respectively by dz/ds and dx/ds and subtracting, we obtain

$$u(d^2x/ds^2\ dz/ds - d^2z/ds^2\ dx/ds) + u_z dx/ds - u_x dz/ds = 0.$$

Using Eq.(70) we have

$$(71)\quad d\psi/ds = V(u_z dx/ds - u_x dz/ds).$$

Introducing the variables

$$(72)\quad Y \equiv (Y_1,Y_2,Y_3,Y_4,Y_5)^T = (x,z,\psi,S,t)^T,$$

and changing in variables s to τ=s/S, the resulting, first order system is

$$\begin{aligned}
Y_1' &= Y_4 \cos Y_3,\\
Y_2' &= Y_4 \sin Y_3,\\
Y_3' &= Y_4 V[u_z \cos Y_3 - u_x \sin Y_3], \qquad 0 \le \tau \le 1\\
Y_4' &= 0,\\
Y_5' &= Y_4 u,
\end{aligned} \tag{73}$$

where now the prime (') denotes differentiation with respect to τ. Obviously the corresponding boundary conditions are:

$$\begin{aligned}
Y_1(0) &= x_0,\\
Y_2(0) &= z_0,\\
Y_1(1) &= x_1,\\
Y_2(1) &= z_1,\\
Y_5(0) &= 0.
\end{aligned} \tag{74}$$

Solving Eq.(73) with the boundary conditions (74) will give us the solution of the two point ray tracing problem in a two dimensional inhomogeneous medium.

6.4 An Alternative Formulation of the Two Point Ray Tracing Problem in 2-D Media

Using x-z coordinates and taking the ray path as a function of the x-coordinate, the differential ray length ds and the travel time T are, respectively:

$$ds = [1+(dz/dx)^2]^{1/2}\, dx, \tag{75}$$

$$T = \int_{x_0}^{x_1} [1+(dz/dx)^2]^{1/2}/V(x,z)\, dx. \tag{76}$$

According to Euler's equations (53) we have:

$$W = [1+(dz/dx)^2]^{1/2}/V(x,z),$$

$$d[\partial W/\partial \dot{z})]/dx - \partial W/\partial z = 0,$$

and finally

$$d^2z/dx^2 = V^{-1}[1+(dz/dx)^2][(\partial V/\partial x)(dz/dx) - \partial V/\partial z]. \tag{77}$$

Introducing two new auxilary parameters Z_1 and Z_2 as follows,

$$Z_1 \equiv z,$$
$$Z_2 \equiv dZ_1/dx = dz/dx, \tag{78}$$

Eq.(77) becomes

$$dZ_1/dx = Z_2,$$

(79) $$dZ_2/dx = V^{-1}(1+Z_2^2)[(\partial V/\partial x)Z_2 - \partial V/\partial Z_1], \qquad x_0 \le x \le x_1$$

with the relative boundary conditions:

(80) $$Z_1(x_0) = z_0,$$
$$Z_1(x_1) = z_1.$$

Solving Eq.(79) with (80) will give the solution of the ray path between two end points $A(x_0,z_0)$ and $B(x_1,z_1)$ in a two dimensional medium.

6.5 Treatment of Interfaces in the Medium

From the above discussion we can see that the seismic ray tracing problem has been reduced to solving a set of first order differential equations of the form

(81) $$\vec{Y}' = \vec{F}(\tau,\vec{Y}(\tau)), \qquad \tau \in [0,1]$$

with the boundary conditions formally written as

$$\vec{g}[\vec{Y}(0),\vec{Y}(1)] = 0,$$

where $\vec{Y}$, $\vec{F}$, $\vec{g}$ are all d - dimensional vector functions, and defined in a R-dimensional space. $\vec{F}$ is a nonlinear function.

In ray tracing problems, as in many other applications, it is possible for problems (64) or (73) to have jump discontinuities within the region of interest. These discontinuities appear in the function $\vec{F}(\tau,\vec{Y})$. V. Pereyra et al. (1980) have discussed this problem. If $\vec{F}$ is discontinuous with respect to the independent variable at a known location δ_1, $0<\delta_1<1$, then the solution $\vec{Y}^*(\tau)$ or some of its

derivatives may be discontinuous at δ_1 as well. If $\vec{F}$ is discontinuous with respect to $\vec{Y}$ then the location of the possible discontinuities in $\vec{Y}^*(\tau)$ or its derivatives are not known a priori.

Assume for simplicity that $\vec{F}$ becomes discontinuous when the ray or trajectory $\vec{Y}^*(\tau)$, traverses a given surface $\Phi(\tau,\vec{Y})=0$ in $(0,1)\in R^d$ and that this occurs only once for $\tau\in(0,1)$. Thus the condition for $\vec{Y}^*(\tau)$ to lie on this surface is that $\Phi(\tau,\vec{Y}^*(\tau))=0$. We will call δ_1 that value of τ for which the ray touches the interface.

Assuming that the desired solution has a known behavior across the discontinuity, i.e. obeys a jump discontinuity condition of the form $\vec{D}[\vec{Y}^*(\delta_1,-),\vec{Y}^*(\delta_1,+)]=0$ where $\vec{D}$ is d - dimensional and "±" represent the value on the right and left of the discontinuity, this problem can be reduced to the form (64) or (73) for some data. Any number of interfaces of this type can be treated simultaneously in the same way, provided one knows the order in which they are traversed and in which region the ray is situated, i.e. which velocity function applies. It is obvious that one can as easily consider both reflected and refracted rays, as well as a change from P to S waves and vice versa.

The idea of the reduction is to consider system (73) independently in two regions $[0,\delta_1]$, $[\delta_1,1]$, and use the interface condition $\vec{D}=0$, to couple these two problems. In each one of the intervals the problem is smooth. In the case of known crossing time δ_1, a system twice the size of the

original one is considered (i.e. one subsystem for each subinterval), adding the interface conditions and the necessary d additional boundary conditions.

The case of an unknown crossing time requires the introduction of an auxiliary dependent variable $\delta_1(\tau)$, which satisfies the trivial first order differential equation $\delta_1'(\tau)=0$. The additional boundary condition is now given by the definition of the interface and the requirement that $\vec{Y}^*$ lie on it for $\tau=\delta_1$:

$$\Phi[\delta_1,\vec{Y}^*(\delta_1)] = 0.$$

If we call $[0,\delta_1]$, $[\delta_1,1]$ as regions (1) and (2), respectively, map each interval time into $[0,1]$ and call $\vec{Y}_{(1)}$ and $\vec{Y}_{(2)}$ the solution on these subintervals, we have an augmented system of the form:

$$\begin{aligned} \vec{Y}_{(1)}' &= \delta_1\vec{F}(\tau,\vec{Y}_{(1)}), \\ \vec{Y}_{(2)}' &= (1-\delta_1)\vec{F}(\tau,\vec{Y}_{(2)}), \qquad \tau \in [0,1] \\ \delta_1' &= 0, \end{aligned} \tag{82}$$

with the boundary conditions:

$$\begin{aligned} \vec{g}[\vec{Y}_{(1)}(0),\vec{Y}_{(2)}(1)] &= 0, \\ \vec{D}[\vec{Y}_{(1)}(1),\vec{Y}_{(2)}(0)] &= 0, \\ \Phi[\delta_1(1),\vec{Y}_{(1)}(1)] &= 0. \end{aligned} \tag{83}$$

Apparently the size of the problem has been greatly increased from d equations to (2*d+1) equations, or in the case of K interfaces to (K+1)*d+K equations. However, the

real dimension of the problem is N*E, E being number of equations and N the number of mesh points.

6.6 A Simple Example for the Treatment of the Interfaces in a 2-D Medium

The theory just developed can now be applied to equation (73). For simplicity, we will start with the case which has only one discontinuity in the velocity field, i.e. when the ray crosses the curve. let

$$\overline{Y} \equiv (Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10})^T, \tag{84}$$

$$\equiv (x_{(1)}, z_{(1)}, \psi_{(1)}, S_{(1)}, t_{(1)}, x_{(2)}, z_{(2)}, \psi_{(2)}, S_{(2)}, t_{(2)})^T.$$

From Eq.(73) and the theory described in section 6.5 we may have the following set of the differential equations:

$$\begin{aligned}
Y_1' &= Y_4 \cos Y_3, \\
Y_2' &= Y_4 \sin Y_3, \\
Y_3' &= Y_4 V_{(1)} (u_{(1)z} \cos Y_3 - u_{(1)x} \sin Y_3), \\
Y_4' &= 0, \\
Y_5' &= Y_4 u_{(1)}, \\
Y_6' &= Y_9 \cos Y_8, \\
Y_7' &= Y_9 \sin Y_8, \\
Y_8' &= Y_9 V_{(2)} (u_{(2)x} \cos Y_8 - u_{(2)x} \sin Y_8), \\
Y_9' &= 0, \\
Y_{10}' &= Y_9 u_{(2)}.
\end{aligned} \qquad 0 \le \tau \le 1 \tag{85}$$

Two new conditions will be given, by Snell's law, which the ray must obey when impinging on an interface separating

two different elastic media, and by the fact that the incident point C has to be on the curve $\Phi(C)=0$ is reached at $\tau=\delta_1$.

Setting $\psi_1(1)$ as the angle of incidence, $\psi_2(0)$ the angle of refraction, and letting θ_1, θ_2 be the angles between the ray direction and the normal direction of the curve, respectively in region (1) and in region (2), and letting η_1, η_2 be the angles between the normal direction and the x axis, respectively in region (1) or (2), we have:

$$\theta_1 = \psi_1(1) - \eta_1.$$

Because of

$$\cos\eta_1 = \Phi_x/(\Phi_x^2 + \Phi_z^2)^{1/2},$$
$$\sin\eta_1 = \Phi_z/(\Phi_x^2 + \Phi_z^2)^{1/2},$$

we have

$$\begin{aligned}\sin\theta_1 &= [\Phi_x\sin\psi_1(1) - \Phi_z\cos\psi_1(1)]/(\Phi_x^2 + \Phi_z^2)^{1/2},\\ \sin\theta_2 &= [\Phi_x\sin\psi_2(0) - \Phi_z\cos\psi_2(0)]/(\Phi_x^2 + \Phi_z^2)^{1/2}.\end{aligned} \tag{86}$$

Employing Snell's law we have:

$$\begin{aligned}&V_{(2)}(c)[\Phi_x\sin\psi_1(1) - \Phi_z\cos\psi_1(1)] =\\ &V_{(1)}(c)[\Phi_x\sin\psi_2(0) - \Phi_z\cos\psi_2(0)].\end{aligned} \tag{87}$$

Then we have the following boundary conditions subject to Eq.(85):

$$Y_1(0) - x_0 = 0,$$
$$Y_2(0) - z_0 = 0,$$

$$
\begin{aligned}
&Y_5(0) = 0,\\
&Y_6(1) - x_1 = 0,\\
&Y_7(1) - z_1 = 0,\\
&Y_{10}(0) = 0,\\
&Y_8(0) - Y_1(1) = 0,\\
&\Phi(Y_1(1), Y_2(1)) = 0,\\
&\Phi(Y_6(0), Y_7(0)) = 0,\\
&V_{(2)}(Y_6(0), Y_7(0))\cdot\\
&\cdot[\Phi_x(Y_1(1), Y_2(1))\sin Y_3(1) - \Phi_z(Y_1(1), Y_2(1))\cos Y_3(1)]\\
&= V_{(1)}(Y_1(1), Y_2(1))\cdot\\
&\cdot[\Phi_x(Y_6(0), Y_7(0))\sin Y_8(0) - \Phi_z(Y_6(0), Y_7(0))\cos Y_8(0)].
\end{aligned}
\tag{88}
$$

7. A Numerical Method for Solving Ray Equations

A numerical method used to solve equations of the form Eq.(81) (of which the ray tracing equations are examples) is based on an adaptive difference solver for nonlinear two point boundary problems described by two mathematicians V. Pereyra and H. B. Keller and a seismologist W. H. K. Lee (1980). The details of the mathematics of this solver can be found by interested readers in the following set of mathematical papers: H. B. Keller (1968,1969,1974); V. Pereyra (1967,1968,1973); M. Lentini and V. Pereyra (1974, 1975, 1977); A. Bjorck and V. Pereyra (1970). Let us simply outline the method. Later we will refer to this solver as Pereyra's method.

Pereyra's method is based on a simple finite difference approximation to $d\vec{Y}/d\tau$ on a mesh with (n+1) points in the interval [0,1]. Consider a mesh π of points $\{\tau_j\}$, $j = 1,2,.....,n+1$, satisfying:

$$(89) \qquad 0 = \tau_1 < \tau_2 < ..... < \tau_{n+1} = 1.$$

In order to change the nonlinear differential equations into nonlinear difference equations (nonlinear algebraic equations), the basic finite difference approximation to Eq.(81) is considered for the trapezoidal rule:

$$\vec{\Phi}_{\pi}(\vec{W})_j \equiv (\vec{W}_{j+1}-\vec{W}_j)/h_j - 1/2[\vec{F}(\tau_{j+1},\vec{W}_{j+1})+\vec{F}(\tau_j,\vec{W}_j)] = 0, \qquad (90)$$

$$j = 1,2,.....,n$$

$$\vec{g}(\vec{W}_1, \vec{W}_{n-1}) = 0.$$

Equations (90) form a system of (n+1)d nonlinear algebraic equations. Using Newton interaction we may change the nonlinear problem (90) into a linear problem, i.e.

$$(91) \quad \vec{W}^{(k+1)} = \vec{W}^{(k)} - [\vec{\Phi}_\pi'(\vec{W}^{(k)})]^{-1}\vec{\Phi}_\pi(\vec{W}^{(k)}),$$

where $\vec{\Phi}_\pi'$ is the Jacobian matrix of $\vec{\Phi}_\pi$:

$$(91) \quad \vec{\Phi}_\pi'(\vec{W}) = \begin{bmatrix} \vec{G}_1, & 0, & \ldots\ldots, & 0, & \vec{G}_{n-1} \\ \vec{S}_1, & \vec{R}_1, & 0, & \ldots\ldots, & 0 \\ 0, & \vec{S}_2, & \vec{R}_2, & \ldots\ldots, & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0, & \ldots\ldots, & 0, & \vec{S}_n, & \vec{R}_n \end{bmatrix}$$

where:

$$\vec{G}_1 = \partial\vec{g}/\partial W_{1,i},$$

$$\vec{G}_{n-1} = \partial\vec{g}/\partial W_{n-1,i},$$

$$\vec{S}_j = -I/h_j - 1/2\ \partial\vec{F}(\tau_j, \vec{W}_j)/\partial W_{j,i},$$

$$\vec{R}_j = I/h_j - 1/2\ \partial\vec{F}(\tau_{j-1}, \vec{W}_{j-1})/\partial W_{j-1,i},$$

$$i = 1,2,\ldots\ldots,d. \qquad j = 2,\ldots\ldots,J.$$

In order to obtain the more accurate approximation we introduce the operators $\vec{S}_\pi^{(k)}$ which are explained in detail in the papers (Pereyra, 1967, 1968; Keller, 1974). Letting $\vec{S}_\pi^{(0)}(u^{(-1)}) \equiv 0$ and solving successively for k = 0,1,2,.....

$$(93) \quad \vec{\Phi}_\pi(\vec{W}^{(k)}) = \vec{S}_\pi^{(k)}(\vec{W}^{(k-1)}),$$

$$\vec{g}(\vec{W}_1, \vec{W}_{n-1}) = 0.$$

After introducing the concept of an asymptotically equidistributing mesh (see Lentini et al, 1977; Pereyra and Sewell, 1975) we can use a nonuniform mesh to increase the calculation efficiency.

Based on the above idea mentioned, I have programmed Pereyra's method to solve two point ray tracing problems like Eq.(73),(79) and (85). Let us first show one of the features of this method, consisting of an automatic procedure for choosing a nonuniform mesh using the following simple example.

Example: Consider the first order scalar equation

$$\begin{aligned} &Y' = -100\, Y(t), \\ &Y(0) = 1. \end{aligned} \qquad t \in [0,1] \tag{94}$$

whose soultion is

$$Y(t) = e^{-100t}.$$

There is a boundary layer of width 1/100 at t=0 (see Fig.2). Using my program, for the accuracy standard of TOL=10^{-3}, the total number of mesh points needed was N=28. The calculation results with the corresponding analytical solution at the same points is listed in Table 1. We can see of the solution that the mesh points are close together in the region of fast parameter variation (0.0 - 0.083), where 11 points are used; while in the open region, (0.083 - 1.0), only 17 mesh points were needed. If we use the uniform mesh for the same

accuracy, we have to use a total of N=129 points.

The variable mesh technique allows the program to "adapt itself" to the particular problem at hand, and thus it produces more detailed computations where they are needed, as in the case of highly curved rays.

FIGURE 2. Example of the nonuniform mesh technique.

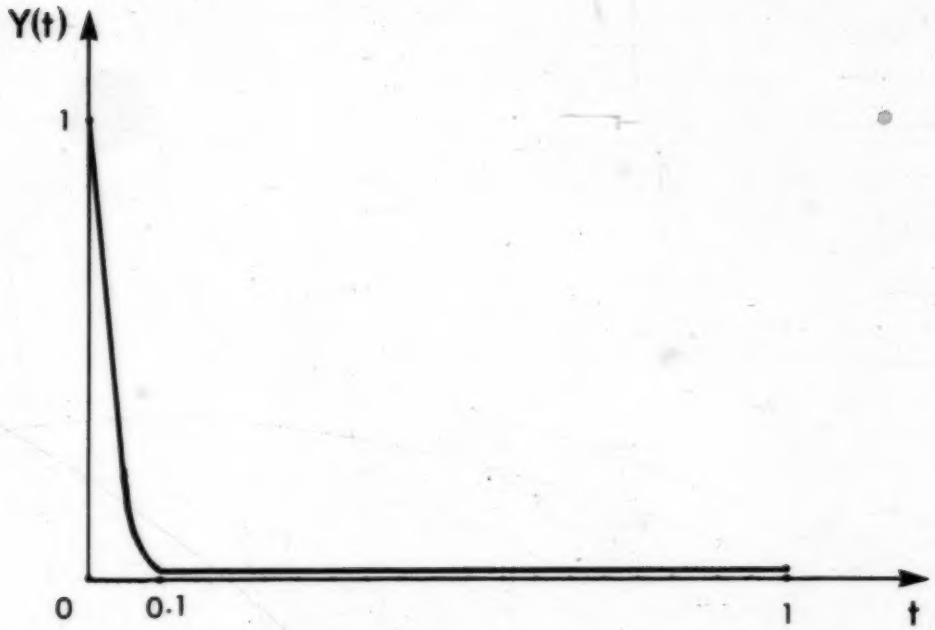

Y(t)
1
0
0.1
1
t

Table 1. Solutions of the Example about the Nonuniform Mesh

N	t	Increment	Analytic Solution	Numerical Solution
1	0.000		1.000	1.000
2	0.007	0.007	0.499	0.499
3	0.014	0.007	0.249	0.250
4	0.021	0.007	0.125	0.125
5	0.028	0.007	0.062	0.063
6	0.035	0.007	0.031	0.031
7	0.042	0.007	0.016	0.016
8	0.049	0.007	0.008	0.008
9	0.056	0.007	0.004	0.004
10	0.063	0.007	0.002	0.002
11	0.083	0.021	0.000	0.000
12	0.104	0.021	0.000	0.000
13	0.125	0.021	0.000	-0.000
14	0.156	0.031	0.000	0.000
15	0.188	0.031	0.000	-0.000
16	0.250	0.063	0.000	0.000
17	0.313	0.063	0.000	-0.000
18	0.375	0.063	0.000	0.000
19	0.438	0.063	0.000	-0.000
20	0.500	0.063	0.000	0.000
21	0.563	0.063	0.000	-0.000
22	0.625	0.063	0.000	0.000
23	0.688	0.063	0.000	-0.000
24	0.750	0.063	0.000	0.000
25	0.813	0.063	0.000	-0.000
26	0.875	0.063	0.000	0.000
27	0.938	0.063	0.000	-0.000
28	1.000	0.063	0.000	0.000

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8. Computer Programs for Two Point Seismic Ray Tracing and Some Examples

As we have shown in sections 6 and 7, the mathematics involved in two point seismic ray tracing problems can become rather complex. Therefore, we must consider how to verify that our computer programs are correct. Traditionally, one tests against some simple cases where analytic solutions are known. It is obvious that this is necessary but not sufficient, as in most cases analytic solutions are known for only trivial cases. If two different programs based on two different methods are written to solve the same problem, then we will have confidence in our programs if they give essentially the same answers.

We have written three computer programs implementing Pereyra's method ZH1 solving Eq.(79) with (80), ZH2 solving Eq.(73) with (74) and ZH3 solving Eq.(85) with (88). These solutions were checked against the programs of W. H. K. Lee ,SRT1 and SRT2, which were based on Wesson's and Lentini's methods, respectively. Thus we have been able to verify that our programs are coded correctly and simultaneously estimate the accuracy and efficiency of Pereyra's method.

8.1 Test Case No.1

In the first case, we assume $V(x,z) = 4 + 0.2\ z$ (km/sec). i.e. velocity is a linear function of only the z-coordinate. We select the two end points as (0, 10) and

(100, 0) (km) to test a ray path similar to the refracted wave case for vertically inhomogeous model. The analytic solution is a circular ray path with its center at (47.5, -20.0) (km). The initial guess is a straight line between the two end points (see Fig.3). The computed solution from program ZH1 together with the analytic solution is shown in Table 3. In Table 2, the computed solutions from the programs SRT1 and SRT2 (Lee's) together with the analytic solution are shown. The maximum deviations of the SRT1 and SRT2 solutions from the analytic solution are about 0.16 km and 0.00028 km respectively; whereas the maximum deviation of the ZH1 solution from the analytic one is about 0.00003 km. So the accuracy of program ZH1 is better than that of programs SRT1 and SRT2.

FIGURE 3. Geometry for case No.1.

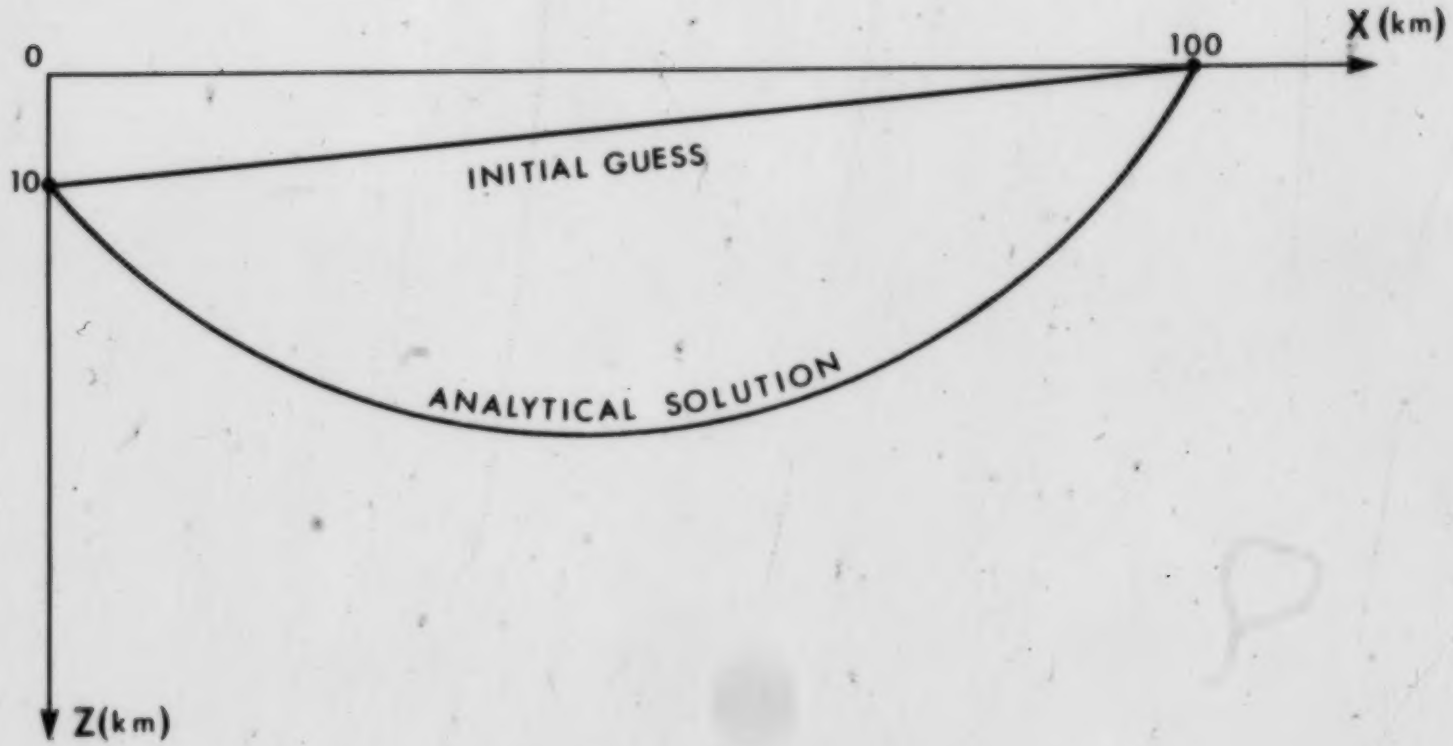
0
100
X (km)
10
INITIAL GUESS
ANALYTICAL SOLUTION
Z(km)

Table 2. Comparison of Analytic, SRT1 and SRT2 Solutions for Case No.1

X (KM)	INITIAL GUESS Z_0 (KM)	ANALYTIC SOLUTION Z_A (KM)	SRT1 SOLUTION Z_1 (KM)	SRT2 SOLUTION Z_2 (KM)	$Z_A - Z_1$ (KM)	$Z_A - Z_2$ (KM)
0.00000	10.00000	10.00000	10.00000	10.00000	0.00000	0.00000
1.00000	9.90000	11.52776	11.51953	11.52774	0.00823	-0.00000
2.00000	9.80000	12.95451	12.94046	12.95450	0.01405	0.00001
3.00000	9.70000	14.29285	14.27466	14.29265	0.01819	-0.00000
4.00000	9.60000	15.55276	15.53099	15.55277	0.02178	-0.00000
5.00000	9.50000	16.74234	16.71777	16.74234	0.02457	0.00000
6.00000	9.40000	17.86818	17.84128	17.86818	0.02691	0.00000
7.00000	9.30000	18.93584	18.91691	18.93584	0.02893	0.00000
8.00000	9.20000	19.94997	19.91925	19.94997	0.03072	0.00000
9.00000	9.10000	20.91454	20.88222	20.91454	0.03232	0.00000
10.00000	9.00000	21.83299	21.79922	21.83299	0.03377	0.00000
11.00000	8.90000	22.70831	22.67319	22.70831	0.03513	0.00000
12.00000	8.80000	23.54308	23.50671	23.54308	0.03636	0.00000
13.00000	8.70000	24.33958	24.30206	24.33958	0.03752	0.00000
14.00000	8.60000	25.09988	25.06123	25.09988	0.03865	0.00000
15.00000	8.50000	25.82574	25.78606	25.82574	0.03969	0.00000
16.00000	8.40000	26.51880	26.47809	26.51861	0.04071	-0.00002
17.00000	8.30000	27.18050	27.13881	27.18050	0.04169	0.00000
18.00000	8.20000	27.81212	27.76950	27.81213	0.04262	-0.00002
19.00000	8.10000	28.41487	28.37138	28.41487	0.04353	0.00000
20.00000	8.00000	28.98979	28.94539	28.98979	0.04440	0.00000
21.00000	7.90000	29.53786	29.49260	29.53786	0.04526	0.00000
22.00000	7.80000	30.05995	30.01384	30.05997	0.04611	-0.00002
23.00000	7.70000	30.55688	30.50996	30.55688	0.04692	0.00000
24.00000	7.60000	31.02939	30.98166	31.02940	0.04773	-0.00002
25.00000	7.50000	31.47815	31.42961	31.47815	0.04854	0.00000
26.00000	7.40000	31.90375	31.85443	31.90375	0.04932	0.00000
27.00000	7.30000	32.30678	32.25670	32.30678	0.05008	0.00000
28.00000	7.20000	32.68776	32.63690	32.68774	0.05086	0.00002
29.00000	7.10000	33.04713	32.99553	33.04713	0.05161	0.00000
30.00000	7.00000	33.38539	33.33301	33.38539	0.05238	0.00000
31.00000	6.90000	33.70288	33.64977	33.70288	0.05312	0.00000
32.00000	6.80000	34.00000	33.94612	33.99998	0.05388	0.00002
33.00000	6.70000	34.27705	34.22244	34.27705	0.05461	0.00000
34.00000	6.60000	34.53438	34.47902	34.53439	0.05536	-0.00002
35.00000	6.50000	34.77225	34.71613	34.77225	0.05612	0.00000
36.00000	6.40000	34.99091	34.93404	34.99091	0.05687	0.00000
37.00000	6.30000	35.19057	35.13295	35.19057	0.05762	0.00000
38.00000	6.20000	35.37146	35.31308	35.37146	0.05838	0.00000
39.00000	6.10000	35.53377	35.47462	35.53375	0.05914	0.00002
40.00000	6.00000	35.67764	35.61771	35.67763	0.05994	0.00002
41.00000	5.90000	35.80322	35.74249	35.80322	0.06073	0.00000
42.00000	5.80000	35.91063	35.84912	35.91063	0.06151	0.00000
43.00000	5.70000	36.00000	35.93767	35.99998	0.06233	0.00002
44.00000	5.60000	36.07138	36.00824	36.07137	0.06314	0.00002
45.00000	5.50000	36.12485	36.06090	36.12485	0.06395	0.00000
46.00000	5.40000	36.16048	36.09569	36.16049	0.06479	0.00000
47.00000	5.30000	36.17828	36.11264	36.17827	0.06564	0.00002
48.00000	5.20000	36.17828	36.11180	36.17827	0.06648	0.00002
49.00000	5.40000	36.16048	36.09312	36.16049	0.06735	0.00000
50.00000	5.00000	36.12485	36.05664	36.12485	0.06821	0.00000
51.00000	4.90000	36.07138	36.00227	36.07137	0.06911	0.00002
52.00000	4.80000	36.00000	35.92998	35.99998	0.07002	0.00002
53.00000	4.70000	35.91063	35.83971	35.91061	0.07092	0.00002
54.00000	4.60000	35.80322	35.73135	35.80321	0.07187	0.00002
55.00000	4.50000	35.67764	35.60481	35.67763	0.07283	0.00002

Table 2. (continue)

X (KM)	INITIAL GUESS Z_0 (KM)	ANALYTIC SOLUTION Z_A (KM)	SRT1 SOLUTION Z_1 (KM)	SRT2 SOLUTION Z_2 (KM)	$Z_A - Z_1$ (KM)	$Z_A - Z$ (KM)
56.00000	4.40000	35.53377	35.45995	35.53375	0.07382	0.00002
57.00000	4.30000	35.37146	35.29665	35.37144	0.07481	0.00002
58.00000	4.20000	35.19057	35.11473	35.19057	0.07584	0.00000
59.00000	4.10000	34.99091	34.91402	34.99088	0.07689	0.00003
60.00000	4.00000	34.77225	34.69431	34.77223	0.07794	0.00002
61.00000	3.90000	34.53438	34.45535	34.53436	0.07903	0.00002
62.00000	3.80000	34.27705	34.19691	34.27704	0.08014	0.00002
63.00000	3.70000	34.00000	33.91872	33.99997	0.08128	0.00003
64.00000	3.60000	33.70288	33.62042	33.70287	0.08246	0.00002
65.00000	3.50000	33.38539	33.30173	33.38536	0.08366	0.00003
66.00000	3.40000	33.04713	32.96225	33.04712	0.08488	0.00002
67.00000	3.30000	32.68776	32.60159	32.68773	0.08617	0.00003
68.00000	3.20000	32.30678	32.21930	32.30675	0.08748	0.00003
69.00000	3.10000	31.90375	31.81491	31.90372	0.08884	0.00003
70.00000	3.00000	31.47815	31.38788	31.47810	0.09027	0.00005
71.00000	2.90000	31.02939	30.93767	31.02936	0.09172	0.00003
72.00000	2.80000	30.55688	30.46364	30.55685	0.09325	0.00003
73.00000	2.70000	30.05995	29.96512	30.05992	0.09483	0.00003
74.00000	2.60000	29.53786	29.44138	29.53783	0.09648	0.00003
75.00000	2.50000	28.98979	28.89160	28.98975	0.09819	0.00005
76.00000	2.40000	28.41487	28.31490	28.41483	0.09998	0.00005
77.00000	2.30000	27.81212	27.71030	27.81207	0.10182	0.00005
78.00000	2.20000	27.18050	27.07674	27.18045	0.10376	0.00005
79.00000	2.10000	26.51880	26.41301	26.51875	0.10579	0.00005
80.00000	2.00000	25.82574	25.71782	25.82570	0.10793	0.00005
81.00000	1.90000	25.09988	24.98969	25.09982	0.11020	0.00006
82.00000	1.80000	24.33958	24.22701	24.33952	0.11258	0.00006
83.00000	1.70000	23.54308	23.42795	23.54301	0.11513	0.00006
84.00000	1.60000	22.70831	22.59048	22.70824	0.11783	0.00008
85.00000	1.50000	21.83299	21.71231	21.83292	0.12068	0.00008
86.00000	1.40000	20.91456	20.79076	20.91446	0.12378	0.00008
87.00000	1.30000	19.94997	19.82289	19.94987	0.12709	0.00009
88.00000	1.20000	18.93584	18.80522	18.93575	0.13062	0.00009
89.00000	1.10000	17.86818	17.73376	17.86809	0.13441	0.00009
90.00000	1.00000	16.74234	16.60382	16.74223	0.13852	0.00011
91.00000	0.90000	15.55276	15.40990	15.55266	0.14287	0.00010
92.00000	0.80000	14.29285	14.14533	14.29273	0.14752	0.00012
93.00000	0.70000	12.95451	12.80222	12.95438	0.15229	0.00013
94.00000	0.60000	11.52776	11.37091	11.52762	0.15685	0.00014
95.00000	0.50000	10.00000	9.83950	9.99984	0.16050	0.00016
96.00000	0.40000	8.35489	8.19316	8.35471	0.16173	0.00018
97.00000	0.30000	6.57065	6.41328	6.57045	0.15737	0.00020
98.00000	0.20000	4.61707	4.47622	4.61679	0.14085	0.00028
99.00000	0.10000	2.44994	2.35161	2.44977	0.09833	0.00017
100.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 3. Computed Solution of ZH1 for Case No.1

X (km)	Analytic Solution Z(A) (km)	ZH1 Solution Z (km)	Z(A) - Z (km)
0.00000	10.00000	10.00000	0.00000
1.00000	11.52777	11.52776	0.00001
2.00000	12.95451	12.95451	0.00000
3.00000	14.29286	14.29285	0.00001
4.00000	15.55278	15.55277	0.00001
5.00000	16.74234	16.74234	0.00000
6.00000	17.86819	17.86818	0.00002
7.00000	18.93585	18.93584	0.00002
8.00000	19.94997	19.94995	0.00002
9.00000	20.91455	20.91454	0.00002
10.00000	21.83301	21.83298	0.00003
11.00000	22.70831	22.70830	0.00002
12.00000	23.54308	23.54306	0.00002
13.00000	24.33960	24.33958	0.00002
14.00000	25.09988	25.09987	0.00002
15.00000	25.82576	25.82573	0.00003
16.00000	26.51881	26.51880	0.00002
17.00000	27.18051	27.18048	0.00003
18.00000	27.81213	27.81210	0.00003
19.00000	28.41487	28.41486	0.00002
20.00000	28.98979	28.98978	0.00002
21.00000	29.53787	29.53784	0.00003
22.00000	30.05997	30.05994	0.00003
23.00000	30.55690	30.55687	0.00003
24.00000	31.02940	31.02937	0.00003
25.00000	31.47815	31.47812	0.00003
26.00000	31.90376	31.90373	0.00003
27.00000	32.30679	32.30676	0.00003
28.00000	32.68776	32.68773	0.00003
29.00000	33.04715	33.04712	0.00003
30.00000	33.38539	33.38536	0.00003
31.00000	33.70288	33.70287	0.00002
32.00000	34.00000	33.99997	0.00003
33.00000	34.27707	34.27704	0.00003
34.00000	34.53439	34.53436	0.00003
35.00000	34.77226	34.77223	0.00003
36.00000	34.99091	34.99088	0.00003
37.00000	35.19058	35.19055	0.00003
38.00000	35.37148	35.37144	0.00003
39.00000	35.53377	35.53374	0.00003
40.00000	35.67764	35.67761	0.00003
41.00000	35.80322	35.80319	0.00003
42.00000	35.91064	35.91061	0.00003
43.00000	36.00000	35.99997	0.00003
44.00000	36.07138	36.07135	0.00003
45.00000	36.12486	36.12483	0.00003
46.00000	36.16049	36.16045	0.00005
47.00000	36.17828	36.17825	0.00003
48.00000	36.17828	36.17825	0.00003

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49.00000	36.16049	36.16045	0.00005
50.00000	36.12486	36.12483	0.00003
51.00000	36.07138	36.07135	0.00003
52.00000	36.00000	35.99997	0.00003
53.00000	35.91064	35.91061	0.00003
54.00000	35.80322	35.80319	0.00003
55.00000	35.67764	35.67761	0.00003
56.00000	35.53377	35.53374	0.00003
57.00000	35.37148	35.37144	0.00003
58.00000	35.19058	35.19055	0.00003
59.00000	34.99091	34.99088	0.00003
60.00000	34.77226	34.77223	0.00003
61.00000	34.53439	34.53436	0.00003
62.00000	34.27707	34.27704	0.00003
63.00000	34.00000	33.99997	0.00003
64.00000	33.70288	33.70285	0.00003
65.00000	33.38539	33.38536	0.00003
66.00000	33.04715	33.04712	0.00003
67.00000	32.68776	32.68773	0.00003
68.00000	32.30679	32.30676	0.00003
69.00000	31.90376	31.90373	0.00003
70.00000	31.47815	31.47812	0.00003
71.00000	31.02940	31.02937	0.00003
72.00000	30.55690	30.55687	0.00003
73.00000	30.05997	30.05994	0.00003
74.00000	29.53787	29.53784	0.00003
75.00000	28.98979	28.98976	0.00003
76.00000	28.41487	28.41484	0.00003
77.00000	27.81213	27.81210	0.00003
78.00000	27.18051	27.18048	0.00003
79.00000	26.51881	26.51878	0.00003
80.00000	25.82576	25.82573	0.00003
81.00000	25.09988	25.09987	0.00002
82.00000	24.33960	24.33957	0.00003
83.00000	23.54308	23.54306	0.00002
84.00000	22.70831	22.70828	0.00003
85.00000	21.83301	21.83298	0.00003
86.00000	20.91455	20.91452	0.00003
87.00000	19.94997	19.94995	0.00002
88.00000	18.93585	18.93582	0.00003
89.00000	17.86819	17.86816	0.00003
90.00000	16.74234	16.74232	0.00002
91.00000	15.55278	15.55276	0.00002
92.00000	14.29286	14.29284	0.00002
93.00000	12.95451	12.95450	0.00001
94.00000	11.52777	11.52775	0.00002
95.00000	10.00000	9.99999	0.00001
96.00000	8.35489	8.35488	0.00001
97.00000	6.57066	6.57065	0.00001
98.00000	4.61707	4.61706	0.00001
99.00000	2.44995	2.44994	0.00001
100.00000	0.00000	-0.00000	0.00000

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8.2 Test Case No.2

In this case we assume $V(x,z) = 4 + 0.01\ x + 0.2\ z$ (km/sec). The two end points are the same as in case No.1, i.e. (0, 10) and (100, 0) (km). The analytical solution is a circular ray path with its center at (47.263683, -22.363159) (km). In Table 4, the computed solution from the programs SRT1 and SRT2 (Lee's) are shown. In Table 5 the computed solution from program ZH1 together with the analytic solution are shown. The maximum deviations of the SRT1 and SRT2 solutions from the analytic solution are about 0.14309 km and 0.00014 km respectively; whereas the maximum deviation of the ZH1 solution from the analytic one is about 0.00005 km. In this example we once again see that the accuracy of program ZH1 is better than that of programs SRT1 and SRT2.

Table 4. Comparison of SRT1 and SRT2 Solutions for Case No.2

X (KM)	INITIAL GUESS Z_0 (KM)	SRT1 SOLUTION Z_1 (KM)	SRT2 SOLUTION Z_2 (KM)	$Z_1 - Z_2$ (KM)
0.00000	10.00000	10.00000	10.00000	0.00000
1.00000	9.90000	11.40590	11.41407	-0.00817
2.00000	9.80000	12.72872	12.74280	-0.01408
3.00000	9.70000	13.97702	13.99555	-0.01853
4.00000	9.60000	15.15790	15.17992	-0.02202
5.00000	9.50000	16.27733	16.30220	-0.02487
6.00000	9.40000	17.34042	17.36765	-0.02722
7.00000	9.30000	18.35146	18.38074	-0.02928
8.00000	9.20000	19.31421	19.34526	-0.03105
9.00000	9.10000	20.23187	20.26453	-0.03265
10.00000	9.00000	21.10732	21.14140	-0.03409
11.00000	8.90000	21.94301	21.97839	-0.03539
12.00000	8.80000	22.74113	22.77774	-0.03661
13.00000	8.70000	23.50365	23.54140	-0.03775
14.00000	8.60000	24.22230	24.27112	-0.03882
15.00000	8.50000	24.92863	24.96848	-0.03984
16.00000	8.40000	25.59407	25.63487	-0.04080
17.00000	8.30000	26.22986	26.27158	-0.04172
18.00000	8.20000	26.83716	26.87976	-0.04260
19.00000	8.10000	27.41701	27.46045	-0.04344
20.00000	8.00000	27.97035	28.01460	-0.04425
21.00000	7.90000	28.49805	28.54311	-0.04506
22.00000	7.80000	29.00089	29.04671	-0.04582
23.00000	7.70000	29.47958	29.52615	-0.04657
24.00000	7.60000	29.93480	29.98213	-0.04733
25.00000	7.50000	30.36716	30.41521	-0.04805
26.00000	7.40000	30.77719	30.82597	-0.04878
27.00000	7.30000	31.16542	31.21492	-0.04950
28.00000	7.20000	31.53232	31.58253	-0.05022
29.00000	7.10000	31.87833	31.92924	-0.05092
30.00000	7.00000	32.20383	32.25545	-0.05162
31.00000	6.90000	32.50919	32.56152	-0.05234
32.00000	6.80000	32.79475	32.84778	-0.05302
33.00000	6.70000	33.06081	33.11453	-0.05373
34.00000	6.60000	33.30765	33.36208	-0.05443
35.00000	6.50000	33.53554	33.59065	-0.05511
36.00000	6.40000	33.74469	33.80051	-0.05582
37.00000	6.30000	33.93532	33.99182	-0.05650
38.00000	6.20000	34.10760	34.16481	-0.05721
39.00000	6.10000	34.26172	34.31963	-0.05791
40.00000	6.00000	34.39783	34.45644	-0.05861
41.00000	5.90000	34.51604	34.57533	-0.05930
42.00000	5.80000	34.61647	34.67648	-0.06001
43.00000	5.70000	34.69920	34.75993	-0.06073
44.00000	5.60000	34.76433	34.82578	-0.06145
45.00000	5.50000	34.81189	34.87408	-0.06219
46.00000	5.40000	34.84195	34.90489	-0.06294
47.00000	5.30000	34.85454	34.91823	-0.06369
48.00000	5.20000	34.84964	34.91409	-0.06445
49.00000	5.10000	34.82726	34.89252	-0.06526
50.00000	5.00000	34.78738	34.85344	-0.06606
51.00000	4.90000	34.73000	34.79684	-0.06685
52.00000	4.80000	34.65501	34.72269	-0.06767
53.00000	4.70000	34.56238	34.63087	-0.06850
54.00000	4.60000	34.45201	34.52136	-0.06935
55.00000	4.50000	34.32381	34.39400	-0.07019

Table 4. (continue)

X (KM)	INITIAL GUESS Z_0 (KM)	SRT1 SOLUTION Z_1 (KM)	SRT2 SOLUTION Z_2 (KM)	$Z_1 - Z_2$ (KM)
56.00000	4.40000	34.17763	34.24870	-0.07108
57.00000	4.30000	34.01334	34.08531	-0.07198
58.00000	4.20000	33.83081	33.90369	-0.07288
59.00000	4.10000	33.62982	33.70363	-0.07381
60.00000	4.00000	33.41020	33.48495	-0.07475
61.00000	3.90000	33.17172	33.24745	-0.07573
62.00000	3.80000	32.91414	32.99084	-0.07671
63.00000	3.70000	32.63719	32.71490	-0.07771
64.00000	3.60000	32.34056	32.41933	-0.07877
65.00000	3.50000	32.02396	32.10379	-0.07983
66.00000	3.40000	31.68701	31.76794	-0.08093
67.00000	3.30000	31.32935	31.41141	-0.08206
68.00000	3.20000	30.95055	31.03375	-0.08321
69.00000	3.10000	30.55016	30.63457	-0.08441
70.00000	3.00000	30.12767	30.21330	-0.08563
71.00000	2.90000	29.68257	29.76949	-0.08691
72.00000	2.80000	29.21426	29.30249	-0.08823
73.00000	2.70000	28.72211	28.81171	-0.08960
74.00000	2.60000	28.20541	28.29643	-0.09102
75.00000	2.50000	27.66344	27.75592	-0.09248
76.00000	2.40000	27.09531	27.18933	-0.09402
77.00000	2.30000	26.50012	26.59576	-0.09564
78.00000	2.20000	25.87689	25.97423	-0.09734
79.00000	2.10000	25.22453	25.32364	-0.09911
80.00000	2.00000	24.54181	24.64278	-0.10097
81.00000	1.90000	23.82738	23.93031	-0.10294
82.00000	1.80000	23.07974	23.18475	-0.10501
83.00000	1.70000	22.29726	22.40446	-0.10721
84.00000	1.60000	21.47803	21.58759	-0.10956
85.00000	1.50000	20.62000	20.73203	-0.11203
86.00000	1.40000	19.72076	19.83543	-0.11467
87.00000	1.30000	18.77765	18.89516	-0.11751
88.00000	1.20000	17.78758	17.90813	-0.12054
89.00000	1.10000	16.74699	16.87080	-0.12381
90.00000	1.00000	15.65183	15.77909	-0.12726
91.00000	0.90000	14.49728	14.62817	-0.13089
92.00000	0.80000	13.27774	13.41233	-0.13459
93.00000	0.70000	11.98652	12.12471	-0.13819
94.00000	0.60000	10.61568	10.75693	-0.14125
95.00000	0.50000	9.15562	9.29861	-0.14299
96.00000	0.40000	7.59470	7.73658	-0.14188
97.00000	0.30000	5.91865	6.05377	-0.13512
98.00000	0.20000	4.10985	4.22722	-0.11737
99.00000	0.10000	2.14627	2.22503	-0.07876
100.00000	0.00000	0.00000	0.00000	0.00000

Table 5. Computed Solution of ZH1 for Case No.2

X (km)	Analytic Solution Z(A) (km)	ZH1 Solution Z (km)	Z(A) - Z (km)
0.00000	10.00000	10.00000	0.00000
1.00000	11.41407	11.41407	0.00000
2.00000	12.74281	12.74280	0.00000
3.00000	13.99555	13.99555	0.00000
4.00000	15.17993	15.17992	0.00001
5.00000	16.30220	16.30220	0.00000
6.00000	17.36766	17.36765	0.00002
7.00000	18.38074	18.38072	0.00002
8.00000	19.34526	19.34525	0.00002
9.00000	20.26453	20.26451	0.00002
10.00000	21.14140	21.14139	0.00002
11.00000	21.97841	21.97839	0.00002
12.00000	22.77776	22.77773	0.00003
13.00000	23.54141	23.54138	0.00003
14.00000	24.27113	24.27110	0.00003
15.00000	24.96848	24.96846	0.00002
16.00000	25.63487	25.63486	0.00002
17.00000	26.27159	26.27156	0.00003
18.00000	26.87976	26.87975	0.00002
19.00000	27.46046	27.46043	0.00003
20.00000	28.01462	28.01459	0.00003
21.00000	28.54311	28.54308	0.00003
22.00000	29.04672	29.04669	0.00003
23.00000	29.52617	29.52614	0.00003
24.00000	29.98213	29.98210	0.00003
25.00000	30.41522	30.41519	0.00003
26.00000	30.82597	30.82596	0.00002
27.00000	31.21494	31.21490	0.00003
28.00000	31.58255	31.58252	0.00003
29.00000	31.92926	31.92923	0.00003
30.00000	32.25546	32.25543	0.00003
31.00000	32.56152	32.56149	0.00003
32.00000	32.84779	32.84776	0.00003
33.00000	33.11455	33.11452	0.00003
34.00000	33.36209	33.36206	0.00003
35.00000	33.59067	33.59064	0.00003
36.00000	33.80051	33.80048	0.00003
37.00000	33.99184	33.99181	0.00003
38.00000	34.16483	34.16479	0.00003
39.00000	34.31964	34.31961	0.00003
40.00000	34.45645	34.45642	0.00003
41.00000	34.57536	34.57533	0.00003
42.00000	34.67650	34.67647	0.00003
43.00000	34.75995	34.75992	0.00003
44.00000	34.82581	34.82576	0.00005
45.00000	34.87411	34.87407	0.00005
46.00000	34.90492	34.90488	0.00005
47.00000	34.91824	34.91821	0.00003
48.00000	34.91412	34.91408	0.00005

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49.00000	34.89253	34.89250	0.00003
50.00000	34.85345	34.85342	0.00003
51.00000	34.79687	34.79683	0.00005
52.00000	34.72272	34.72267	0.00005
53.00000	34.63091	34.63087	0.00003
54.00000	34.52138	34.52135	0.00003
55.00000	34.39403	34.39400	0.00003
56.00000	34.24873	34.24869	0.00005
57.00000	34.08534	34.08530	0.00005
58.00000	33.90372	33.90367	0.00005
59.00000	33.70366	33.70361	0.00005
60.00000	33.48499	33.48495	0.00003
61.00000	33.24747	33.24744	0.00003
62.00000	32.99088	32.99084	0.00003
63.00000	32.71494	32.71490	0.00003
64.00000	32.41936	32.41933	0.00003
65.00000	32.10382	32.10379	0.00003
66.00000	31.76797	31.76793	0.00005
67.00000	31.41144	31.41139	0.00005
68.00000	31.03378	31.03375	0.00003
69.00000	30.63458	30.63455	0.00003
70.00000	30.21335	30.21332	0.00003
71.00000	29.76952	29.76949	0.00003
72.00000	29.30254	29.30249	0.00005
73.00000	28.81174	28.81171	0.00003
74.00000	28.29648	28.29643	0.00005
75.00000	27.75595	27.75592	0.00003
76.00000	27.18936	27.18933	0.00003
77.00000	26.59581	26.59578	0.00003
78.00000	25.97427	25.97424	0.00003
79.00000	25.32368	25.32365	0.00003
80.00000	24.64282	24.64279	0.00003
81.00000	23.93036	23.93033	0.00003
82.00000	23.18480	23.18478	0.00002
83.00000	22.40451	22.40448	0.00003
84.00000	21.58763	21.58760	0.00003
85.00000	20.73207	20.73206	0.00002
86.00000	19.83549	19.83548	0.00002
87.00000	18.89522	18.89520	0.00002
88.00000	17.90819	17.90817	0.00002
89.00000	16.87088	16.87085	0.00003
90.00000	15.77917	15.77914	0.00003
91.00000	14.62825	14.62823	0.00002
92.00000	13.41241	13.41239	0.00002
93.00000	12.12479	12.12478	0.00002
94.00000	10.75702	10.75700	0.00002
95.00000	9.29871	9.29869	0.00002
96.00000	7.73669	7.73668	0.00001
97.00000	6.05389	6.05388	0.00001
98.00000	4.22736	4.22736	0.00001
99.00000	2.22512	2.22511	0.00001
100.00000	0.00000	-0.00000	0.00000

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8.3 Summary of Test Case Results

In Table 6 the main results of our test cases are summarized using some solutions of Lee's. We shall show the maximal deviation from the analytic solutions or from each other and the total travel time given by each program. It is clear from Table 6 that all programs SRT1, SRT2 and ZH1 will compute the travel time correctly. Finally, by comparison with analytic solutions, we note that the accuracy of program ZH1 is better than those of programs SRT1 and SRT2. In other words, Pereyra's method is more accurate than Wesson's and Lentini's methods.

Table 6. Comparison of Analytic, SRT1, SRT2 and ZH1 Solutions

Case No.	Velocity Model (km/sec)	Solution	Travel Time (sec)	Maximal Deviation (km)	
1.	4 + 0.2 Z	Analytic	14.6640		
	A(0, 10)	SRT1	14.6625	0.16	(SRT1 - Anal.)
	B(100, 0)	SRT2	14.6624	0.00028	(SRT2 - Anal.)
		ZH1	14.6635	0.00003	(ZH1 - Anal.)
2.	4 + 0.01 X + 0.2 Z	Analytic	13.666		
	A(0, 10)	SRT1	13.666	0.14309	(SRT1 - Anal.)
	B(100, 0)	SRT2	13.666	0.00014	(SRT2 - Anal.)
		ZH1	13.666	0.00005	(ZH1 - Anal.)

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8.4 The Program ZH1 and 5 Computed Examples of Ray Tracing Problems in 2-D Media

In 8.3 section we have compared the results of program ZH1 with Lee's programs SRT1 and SRT2. In order to show the computed efficiency of ZH1 we have used it on MTS computer to compute results using 5 different geological models. The CPU times needed are shown in Table 7. In the Tables 8 and 12 the final output solutions including the velocity model, the total mesh number N used, and the expected accuracy TOL, are all given. We shall see from Table 7 that the computation efficiency of ZH1 is high reasonably and program ZH1 may be used as a useful subroutine in any large program of ray approximation which needs to compute the ray path and the travel time.

The program ZH1 is listed in Appendix I and is written in ALGOLW language and is valid on the MTS computer at the University of Alberta. A user will be asked only to input the velocity model and the desired accuracy.

Table 7. Main Computed Solutions Using ZH1 Program

No.	Velocity Model (km/sec)	Total Travel Time (sec)	Total Ray Length (km)	CPU Time on MTS (sec)
1.	4 + 0.01 X + 0.2 Z A(0, 10); B(60, 20)	7.687	65.828	0.102
2.	4 + 0.01 X + 0.2 Z A(0, 0); B(50, 0)	10.018	56.830	0.134
3.	5 + 0.05 X + 0.1 Z A(20, 20); B(60, 0)	5.503	45.411	0.099
4.	4 + 0.01 X + 0.05 Z A(30, 15); B(60, 5)	6.362	31.758	0.068
5.	6 + 0.05 X + 0.1 Z A(5, 3); B(65, 10)	7.536	61.580	0.104

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Table 8 . Computed Solution of ZH1 for Model 1.

VELOCITY MODEL:

V=Z0+Z1*X+Z2*Z
Z0= 4.000000 Z1= 0.0100000 Z2= 0.2000000

X(0)= 0 Z(0)= 10.00000
X(1)= 60.00000 Z(1)= 20.00000

N= 21
TOL= 1.00'-03 (7.67'-05)

N= 1	X=	0.000	Z=	10.000	Q=	48.646	T=	0.000	S=	0.000
N= 2	X=	3.000	Z=	13.114	Q=	43.496	T=	0.684	S=	4.324
N= 3	X=	6.000	Z=	15.734	Q=	38.755	T=	1.258	S=	8.307
N= 4	X=	9.000	Z=	17.856	Q=	34.310	T=	1.760	S=	12.040
N= 5	X=	12.000	Z=	19.845	Q=	30.089	T=	2.209	S=	15.586
N= 6	X=	15.000	Z=	21.445	Q=	26.042	T=	2.621	S=	18.985
N= 7	X=	18.000	Z=	22.786	Q=	22.130	T=	3.003	S=	22.272
N= 8	X=	21.000	Z=	23.891	Q=	18.324	T=	3.364	S=	25.469
N= 9	X=	24.000	Z=	24.778	Q=	14.600	T=	3.708	S=	28.597
N= 10	X=	27.000	Z=	25.458	Q=	10.938	T=	4.040	S=	31.673
N= 11	X=	30.000	Z=	25.940	Q=	7.321	T=	4.362	S=	34.712
N= 12	X=	33.000	Z=	26.230	Q=	3.733	T=	4.678	S=	37.726
N= 13	X=	36.000	Z=	26.332	Q=	0.160	T=	4.991	S=	40.727
N= 14	X=	39.000	Z=	26.247	Q=	-3.413	T=	5.303	S=	43.728
N= 15	X=	42.000	Z=	25.974	Q=	-6.999	T=	5.615	S=	46.741
N= 16	X=	45.000	Z=	25.509	Q=	-10.613	T=	5.932	S=	49.777
N= 17	X=	48.000	Z=	24.847	Q=	-14.270	T=	6.256	S=	52.849
N= 18	X=	51.000	Z=	23.980	Q=	-17.987	T=	6.589	S=	55.972
N= 19	X=	54.000	Z=	22.894	Q=	-21.785	T=	6.935	S=	59.162
N= 20	X=	57.000	Z=	21.575	Q=	-25.686	T=	7.299	S=	62.439
N= 21	X=	60.000	Z=	20.000	Q=	-29.720	T=	7.687	S=	65.828

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Table 9. Computed Solution of ZH1 for Model 2

VELOCITY MODEL:

V=Z0+Z1*X+Z2*Z
Z0= 4.000000 Z1= 0.0100000 Z2= 0.2000000

X(0)= 0 Z(0)= 0
X(1)= 50.00000 Z(1)= 0

N= 21
TOL= '1.00'-03 (4.86'-04)

N= 1	X=	0.000	Z=	0.000	Q=	49.640	T=	0.000	S=	0.000
N= 2	X=	2.500	Z=	2.631	Q=	43.294	T=	0.849	S=	3.630
N= 3	X=	5.000	Z=	4.761	Q=	37.559	T=	1.536	S=	6.914
N= 4	X=	7.500	Z=	6.505	Q=	32.233	T=	2.124	S=	9.962
N= 5	X=	10.000	Z=	7.932	Q=	27.204	T=	2.644	S=	12.840
N= 6	X=	12.500	Z=	9.087	Q=	22.394	T=	3.118	S=	15.594
N= 7	X=	15.000	Z=	10.000	Q=	17.745	T=	3.558	S=	18.256
N= 8	X=	17.500	Z=	10.693	Q=	13.213	T=	3.974	S=	20.850
N= 9	X=	20.000	Z=	11.178	Q=	8.765	T=	4.374	S=	23.397
N= 10	X=	22.500	Z=	11.466	Q=	4.370	T=	4.762	S=	25.913
N= 11	X=	25.000	Z=	11.561	Q=	0.000	T=	5.145	S=	28.415
N= 12	X=	27.500	Z=	11.466	Q=	-4.370	T=	5.526	S=	30.917
N= 13	X=	30.000	Z=	11.178	Q=	-8.765	T=	5.910	S=	33.433
N= 14	X=	32.500	Z=	10.693	Q=	-13.213	T=	6.302	S=	35.980
N= 15	X=	35.000	Z=	10.000	Q=	-17.744	T=	6.707	S=	38.574
N= 16	X=	37.500	Z=	9.087	Q=	-22.394	T=	7.131	S=	41.236
N= 17	X=	40.000	Z=	7.932	Q=	-27.204	T=	7.584	S=	43.990
N= 18	X=	42.500	Z=	6.505	Q=	-32.233	T=	8.075	S=	46.868
N= 19	X=	45.000	Z=	4.761	Q=	-37.559	T=	8.623	S=	49.917
N= 20	X=	47.500	Z=	2.631	Q=	-43.294	T=	9.254	S=	53.201
N= 21	X=	50.000	Z=	-0.000	Q=	-49.640	T=	10.018	S=	56.830

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Table 10. Computed Solution of ZH1 for Model 3.

VELOCITY MODEL:

V=ZO+Z1*X+Z2*Z
ZO= 5.000000 Z1= 0.0500000 Z2= 0.1000000

X(0)= 20.00000 Z(0)= 20.00000
X(1)= 60.00000 Z(1)= 0

N= 21
TOL= 1.00'-03 (3.87'-05)

N= 1	X=	20.000	Z=	20.000	Q=	-9.211	T=	0.000	S=	0.000
N= 2	X=	22.000	Z=	19.648	Q=	-10.763	T=	0.253	S=	2.031
N= 3	X=	24.000	Z=	19.239	Q=	-12.323	T=	0.505	S=	4.072
N= 4	X=	26.000	Z=	18.774	Q=	-13.893	T=	0.757	S=	6.126
N= 5	X=	28.000	Z=	18.250	Q=	-15.473	T=	1.009	S=	8.193
N= 6	X=	30.000	Z=	17.666	Q=	-17.066	T=	1.262	S=	10.277
N= 7	X=	32.000	Z=	17.021	Q=	-18.672	T=	1.515	S=	12.378
N= 8	X=	34.000	Z=	16.314	Q=	-20.293	T=	1.770	S=	14.499
N= 9	X=	36.000	Z=	15.541	Q=	-21.932	T=	2.027	S=	16.643
N= 10	X=	38.000	Z=	14.702	Q=	-23.588	T=	2.287	S=	18.812
N= 11	X=	40.000	Z=	13.794	Q=	-25.268	T=	2.549	S=	21.009
N= 12	X=	42.000	Z=	12.813	Q=	-26.971	T=	2.815	S=	23.236
N= 13	X=	44.000	Z=	11.757	Q=	-28.699	T=	3.085	S=	25.498
N= 14	X=	46.000	Z=	10.622	Q=	-30.457	T=	3.360	S=	27.798
N= 15	X=	48.000	Z=	9.403	Q=	-32.247	T=	3.640	S=	30.140
N= 16	X=	50.000	Z=	8.097	Q=	-34.073	T=	3.927	S=	32.529
N= 17	X=	52.000	Z=	6.696	Q=	-35.938	T=	4.222	S=	34.970
N= 18	X=	54.000	Z=	5.195	Q=	-37.851	T=	4.525	S=	37.471
N= 19	X=	56.000	Z=	3.585	Q=	-39.813	T=	4.838	S=	40.039
N= 20	X=	58.000	Z=	1.857	Q=	-41.833	T=	5.164	S=	42.681
N= 21	X=	60.000	Z=	0.000	Q=	-43.919	T=	5.503	S=	45.411

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Table 11. Computed Solution of ZH1 for Model 4.

VELOCITY MODEL:

V=Z0+Z1*X+Z2*Z
Z0= 4.000000 Z1= 0.0100000 Z2= 0.0500000

X(0)= 30.00000 Z(0)= 15.00000
X(1)= 60.00000 Z(1)= 5.000000

N= 21
TOL= 1.00'-03 (2.19'-04)

N= 1	X=	30.000	Z=	15.000	Q=	-9.252	T=	0.000	S=	0.000
N= 2	X=	31.500	Z=	14.744	Q=	-10.132	T=	0.301	S=	1.522
N= 3	X=	33.000	Z=	14.464	Q=	-11.014	T=	0.603	S=	3.048
N= 4	X=	34.500	Z=	14.160	Q=	-11.899	T=	0.906	S=	4.578
N= 5	X=	36.000	Z=	13.831	Q=	-12.787	T=	1.210	S=	6.114
N= 6	X=	37.500	Z=	13.479	Q=	-13.678	T=	1.515	S=	7.655
N= 7	X=	39.000	Z=	13.101	Q=	-14.573	T=	1.822	S=	9.201
N= 8	X=	40.500	Z=	12.699	Q=	-15.470	T=	2.130	S=	10.754
N= 9	X=	42.000	Z=	12.271	Q=	-16.372	T=	2.439	S=	12.314
N= 10	X=	43.500	Z=	11.817	Q=	-17.278	T=	2.751	S=	13.881
N= 11	X=	45.000	Z=	11.337	Q=	-18.189	T=	3.065	S=	15.456
N= 12	X=	46.500	Z=	10.831	Q=	-19.104	T=	3.380	S=	17.039
N= 13	X=	48.000	Z=	10.298	Q=	-20.025	T=	3.699	S=	18.631
N= 14	X=	49.500	Z=	9.738	Q=	-20.951	T=	4.020	S=	20.232
N= 15	X=	51.000	Z=	9.149	Q=	-21.882	T=	4.344	S=	21.844
N= 16	X=	52.500	Z=	8.532	Q=	-22.820	T=	4.671	S=	23.466
N= 17	X=	54.000	Z=	7.887	Q=	-23.765	T=	5.001	S=	25.099
N= 18	X=	55.500	Z=	7.211	Q=	-24.716	T=	5.335	S=	26.744
N= 19	X=	57.000	Z=	6.505	Q=	-25.674	T=	5.673	S=	28.402
N= 20	X=	58.500	Z=	5.769	Q=	-26.641	T=	6.015	S=	30.073
N= 21	X=	60.000	Z=	5.000	Q=	-27.615	T=	6.362	S=	31.758

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Table 12. Computed Solution of ZH1 for Model 5.

VELOCITY MODEL:

V=ZO+Z1*X+Z2*Z
ZO= 5.599999 Z1= 0.0500000 Z2= 0.1000000

X(0)= 5.000000 Z(0)= 3.000000
X(1)= 65.00000 Z(1)= 10.00000

N= 21
TOL= 1.00'-03 (1.51'-05)

N= 1	X=	5.000	Z=	3.000	Q=	26.104	T=	0.000	S=	0.000
N= 2	X=	8.000	Z=	4.403	Q=	24.012	T=	0.526	S=	3.312
N= 3	X=	11.000	Z=	5.675	Q=	21.953	T=	1.021	S=	6.570
N= 4	X=	14.000	Z=	6.823	Q=	19.924	T=	1.490	S=	9.782
N= 5	X=	17.000	Z=	7.851	Q=	17.921	T=	1.936	S=	12.954
N= 6	X=	20.000	Z=	8.765	Q=	15.940	T=	2.363	S=	16.090
N= 7	X=	23.000	Z=	9.566	Q=	13.979	T=	2.772	S=	19.195
N= 8	X=	26.000	Z=	10.259	Q=	12.034	T=	3.166	S=	22.274
N= 9	X=	29.000	Z=	10.846	Q=	10.103	T=	3.546	S=	25.331
N= 10	X=	32.000	Z=	11.329	Q=	8.183	T=	3.915	S=	28.369
N= 11	X=	35.000	Z=	11.709	Q=	6.273	T=	4.274	S=	31.393
N= 12	X=	38.000	Z=	11.989	Q=	4.370	T=	4.624	S=	34.406
N= 13	X=	41.000	Z=	12.168	Q=	2.471	T=	4.966	S=	37.412
N= 14	X=	44.000	Z=	12.248	Q=	0.576	T=	5.302	S=	40.413
N= 15	X=	47.000	Z=	12.228	Q=	-1.319	T=	5.632	S=	43.413
N= 16	X=	50.000	Z=	12.110	Q=	-3.216	T=	5.956	S=	46.415
N= 17	X=	53.000	Z=	11.891	Q=	-5.116	T=	6.277	S=	49.423
N= 18	X=	56.000	Z=	11.572	Q=	-7.022	T=	6.595	S=	52.440
N= 19	X=	59.000	Z=	11.152	Q=	-8.935	T=	6.910	S=	55.469
N= 20	X=	62.000	Z=	10.628	Q=	-10.859	T=	7.224	S=	58.515
N= 21	X=	65.000	Z=	10.000	Q=	-12.795	T=	7.536	S=	61.580

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8.5 A Simple Computed Example for 2-D Model With an Interface

We have programed Eq.(85) with the related boundary conditions (88) which expresses two point ray tracing problem in a 2-D model with an interface. Using this program we computed quantities for the following model.

$V_1(x,z) = 8.5 + 0.04\ x + 0.05\ z$ (km/sec),

$V_2(x,z) = 3.0 + 0.05\ x + 0.2\ z$ (km/sec),

A(0,60) and B(30,0) (km),

Interface: $z = 50$ (km).

The final computed solutions are:

(1) The position of incident point C is:
($x = 7.978$ km, $z = 50.000$ km).

(2) The incident angle is:
$\theta_i = 36.36°$.

(3) The reflected angle is:
$\theta_r = 44.51°$.

(4) The length of path in region (1) is:
$S_{(1)} = 12.795$ km.

(5) The length of path in region (2) is:
$S_{(2)} = 55.889$ km.

(6) The total length of path is:
$S = S_{(1)} + S_{(2)} = 68.684$ km.

(7) The travel time in region (1) is:

$T_{(1)} = 1.121$ sec.

(8) The travel time in region (2) is:

$T_{(2)} = 6.531$ sec.

(9) The Total travel time is:

$T = T_{(1)} + T_{(2)} = 7.652$ sec.

Using the incident angle, reflected angle, $V_1(c) = 13.399$ km/sec and $V_2(c) = 11.319$ km/sec we have:

$$\sin\theta_1/V_1(c) = 0.0527 = \sin\theta_2/V_2(c),$$

which shows us that the computed solutions satisfy Snell's law at the incident point c. This means that the solution computed by our program is correct.

9. Method for Computation of Ray Amplitudes

From Eq.(40) or (42) we can see that the most complicated part in the computation of ray amplitudes is the evaluation of the geometrical spreading, which can be simply expressed in terms of the Jacobian, J, of the transformation from Cartesian coordinates (x,y,z) into ray coordinates (s,q_2,q_3).

In what follows, we shall be interested mainly in the evaluation of the spreading function J, which can be computed in several ways. V. Cerveny et al (1979) have summarized some methods which are based on the computation of the Jacobian, J, at a number of points along the ray. The second or third derivative of the velocity is usually required in those schemes. It is obvious that the numerical computations can become rather tedious using the above methods.

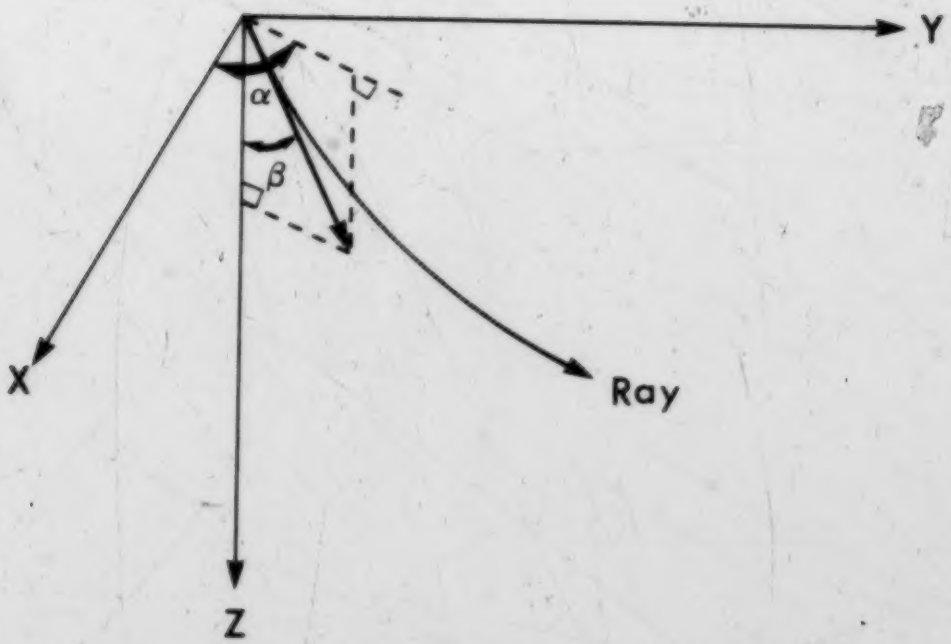
Here we shall present a new method of calculating the Jacobian, J. We are interested in solving J based on the ray tracing system Eq.(63) - (66). As shown in Fig.4, taking q_2 and q_3 as the taking off angle β and the azimuth angle α respectively, we have:

$$(95) \quad \Delta\beta = \mp \Delta(dz/ds) \,/\, (1 - (dz/ds)^2)^{1/2},$$

$$(96) \quad \Delta\alpha = [(dx/ds)\Delta(dy/ds) - (dy/ds)\Delta(dx/ds)] \,/\, [(dx/ds)^2 + (dy/ds)^2],$$

where, "Δ" indicates the small increase.

FIGURE 4. Definition of the taking off angle β and the azimuth angle α.


Y
α
β
X
Ray
Z

We may introduce the following perturbations in two different fashions.

At the taking off point (source) of the ray under consideration, we increase the quantity dz/ds by the small increment $\Delta(dz/ds)$ and as a result (dx/ds) and (dy/ds) become $\lambda(dx/ds)$ and $\lambda(dy/ds)$, respectively, where, λ is determined according to Eq.(65) as:

$$(97) \quad [dz/ds + \Delta(dz/ds)]^2 + \lambda^2[(dx/ds)^2 + (dy/ds)^2] = 1.$$

This only produces a change in the angle β.

At the taking off point, we introduce the small incremental quntities $\Delta(dx/ds)$ and $\Delta(dy/ds)$ into (dx/ds) and (dy/ds), respectively, which gives:

$$(98) \quad \begin{aligned} &\Delta(dy/ds) = \lambda\Delta(dx/ds), \\ &[dx/ds + \Delta(dx/ds)]^2 + [dy/ds + \lambda\Delta(dx/ds)]^2 \\ &\qquad = (dx/ds)^2 + (dy/ds)^2. \end{aligned}$$

This only produces a change in the angle α.

Upon solving the ray tracing system Eq.(63)-(66), We also get at the same time the following values at the source: dx(0)/ds, dy(0)/ds, dz(0)/ds. If we change the values of dx(0)/ds, dy(0)/ds, dz(0)/ds by introducing the small perturbations as mentioned above. we will get two new groups of initial values $dx(0)/ds|_i$, $dy(0)/ds|_i$ and $dz(0)/ds|_i$ (i = 2 or 3) and form the following initial value problems which describes the perturbed rays using the same symbols as in Eq.(63) - (66):

$$
\begin{aligned}
&dY_1/ds = Y_2,\\
&dY_2/ds = V[u_x - GY_2],\\
&dY_3/ds = Y_4,\\
&dY_4/ds = V[u_y - GY_4],\\
&dY_5/ds = Y_6,\\
&dY_6/ds = V[u_z - GY_6],
\end{aligned}
\tag{99}
$$

with the related initial value conditions:

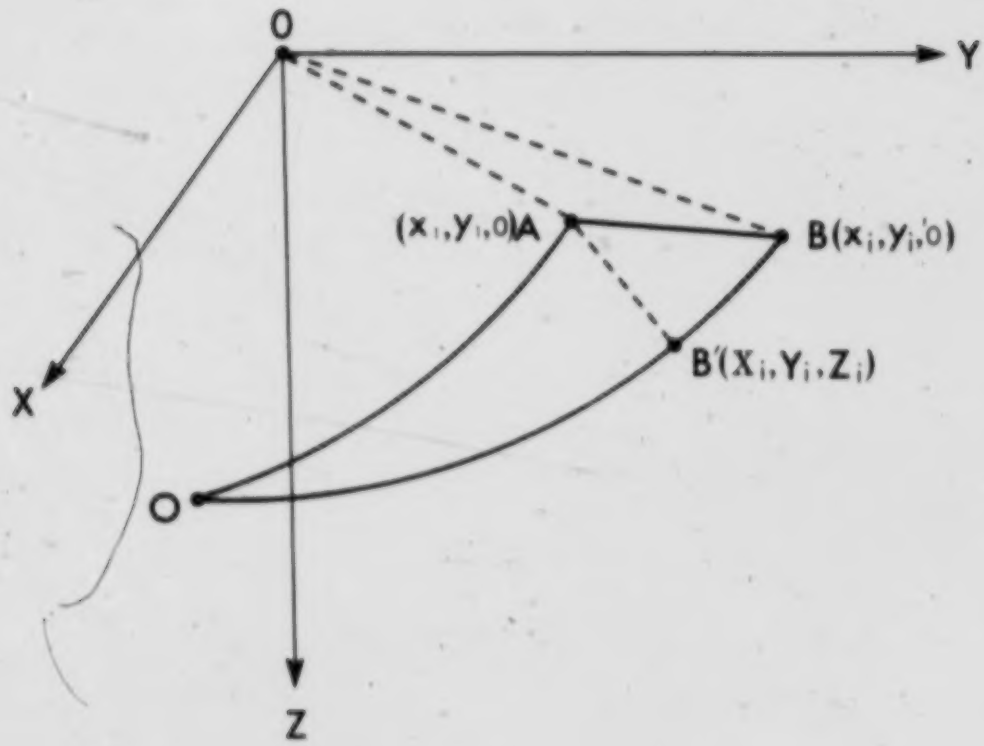
$$
\begin{aligned}
&Y_1(0) = x_0, \quad Y_2(0) = dx(0)/ds|_i,\\
&Y_3(0) = y_0, \quad Y_4(0) = dy(0)/ds|_i, \qquad (i = 2 \text{ or } 3)\\
&Y_5(0) = z_0, \quad Y_6(0) = dz(0)/ds|_i.
\end{aligned}
\tag{100}
$$

Solving Eq.(99) and (100) we can get the coordinates of the perturbed rays at the surface $(x_i, y_i, 0)$, where, $i = 2$ or 3.

As shown in Fig.5, the point O denotes the source and OA is the ray under consideration while OB_i $(i=2,3)$ are the perturbed rays. The points A and B_i $(i=2,3)$ are all placed at the surface, with $A(x_1, y_1, 0)$ being the receiver point.

It is obvious that the point $B(x_i, y_i, 0)$, where $i = 2$ or 3, is not usually placed on the wavefront plane which passes through the receiver point A.

FIGURE 5. The ray under consideration OA and the perturbed ray OB.


O
Y
X
Z
O
$(x_i, y_i, 0)$A
$B(x_i, y_i, 0)$
$B'(X_i, Y_i, Z_i)$

In the following, using elementary geometrical reasoning we shall consider the coordinate of the point B', $X(S)_i$, $Y(S)_i$, $Z(S)_i$ $(i = 2 \text{ or } 3)$, where the perturbed ray intersects with the wavefront plane which passes through the receiver point, A.

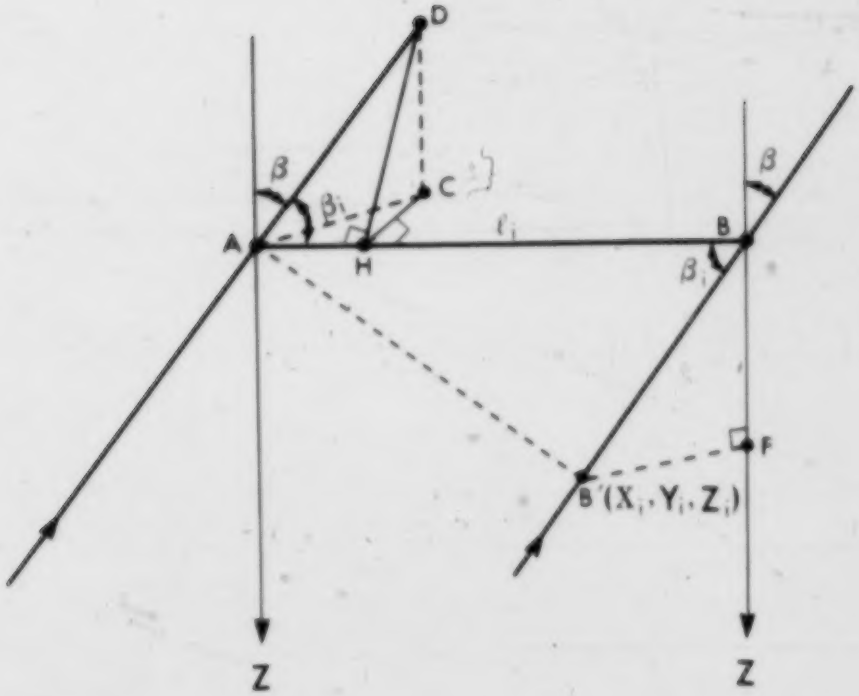
Writing $|AB| = \ell_i$ we have:

$$\ell_i = [(x_i - x_1)^2 + (y_i - y_1)^2]^{1/2}, \qquad i = 2 \text{ or } 3. \tag{101}$$

As shown in Fig.6, at point A, let's define D so that $|AD| = 1$ along the tangential direction to the ray OA and assume β is the acute angle between AD and the z-axis. Making DC parallel to the z-axis and CH perpendicular to AB, we then have DH perpendicular to AB.

Assume $\beta+d\beta$ is the acute angle between the perturbed ray OB and z-coordinate. Although $d\beta$ is not usually equal to zero, we may neglect it because its contribution is a small quantity of second order. So we may consider that AB, AD and BB' are all placed on the same plane and AB' is perpendicular to BB'.

FIGURE 6. Explanation of various symbols for the computation of the quantity of BB'.


D
C
β
β₁
A
H
ℓ₁
B
β
β₁
F
B'(X₁, Y₁, Z₁)
Z
Z

As shown in Fig.7, letting $\alpha(AB)$ be the angle between AB and x-coordinate, we have:

$$\tan\alpha(AB) = (y_i-y_1)/(x_i-x_1), \qquad i = 2 \text{ or } 3. \tag{102}$$

By definition of α we may write the angle between CA and x-coordinate as α_1. If we denote the angle between CA and AB as α_i, we have:

$$\begin{aligned} &\alpha_i = \alpha_1 - \alpha(AB), \\ &\tan\alpha_i = [\tan\alpha_1-\tan\alpha(AB)] \,/\, [1+\tan\alpha_1\tan\alpha(AB)]. \end{aligned} \tag{103}$$

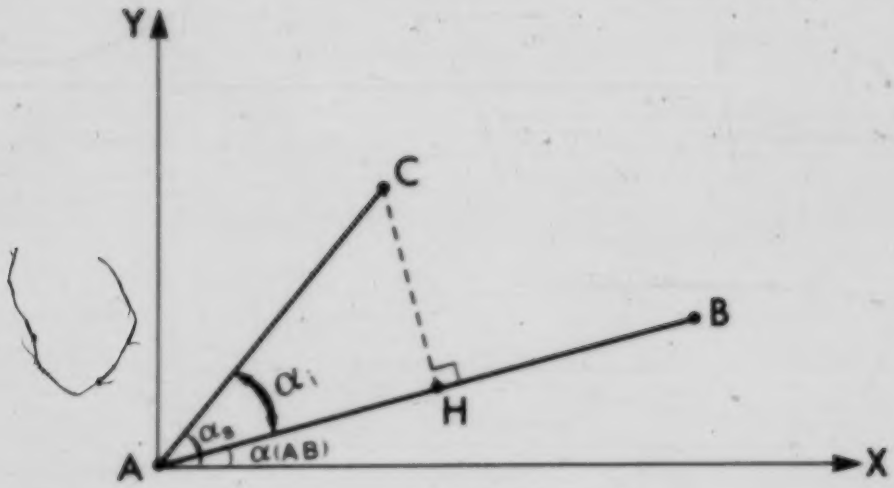
Assuming the angle between DA and AH is β_i, we have:

$$\begin{aligned} &\cos\beta_i = |AH|/|AD| = |AC|\cos\alpha_i = \sin\beta\cos\alpha_i, \\ &|BB'| = \ell_i\cos\beta_i = \ell_i\sin\beta\cos\alpha_i. \end{aligned} \qquad i = 2 \text{ or } 3, \tag{104}$$

From Fig.6 it is obvious that :

$$Z_i(S) = BF = |BB'|\cos\beta = 2^{-1}\ell_i\sin2\beta\cos\alpha_i. \tag{105}$$

FIGURE 7. Definitions of the angles $\alpha(AB)$ and α_i and determination of the coordinate Z_i.


Y
C
B
H
α_1
α_2
$\alpha_{(AB)}$
A
X

As shown in Fig.8, because B'K is perpendicular to the x-y plane we have:

$$|BK| = |BB'|\sin\beta = \ell_1 \sin^2\beta\cos\alpha_1,$$

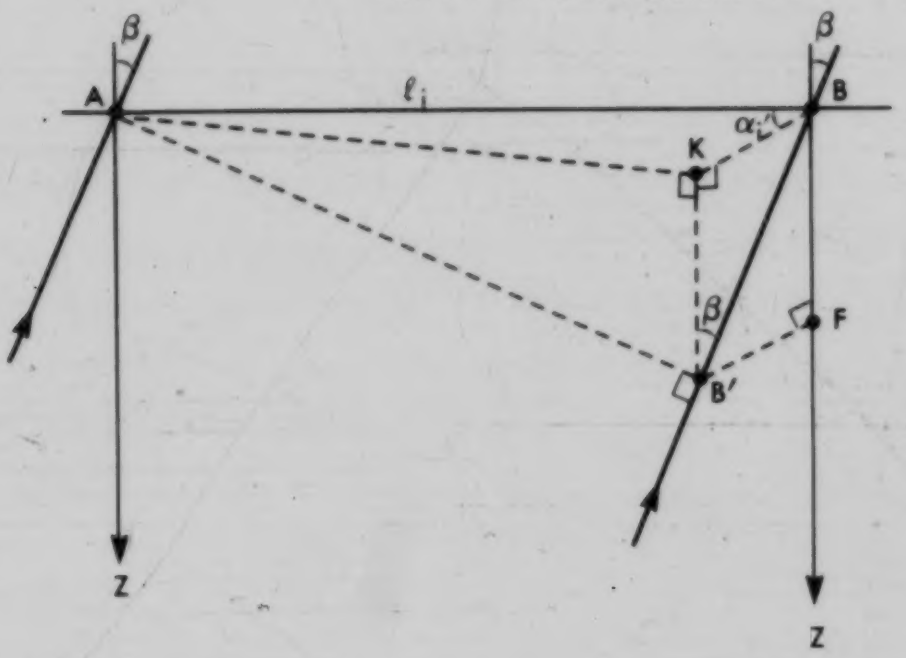
(106)

$$|AK|^2 = |AB'|^2 - |B'K|^2 = \ell_1^2 - |BB'|^2 - Z_1^2.$$

Letting the angle between BK and AB be α_1' we have:

(107) $$\cos\alpha_1' = (|BK|^2 + \ell_1^2 - |AK|^2) / (2\ell_1|BK|)$$
$$= [\cos^2\alpha_1\sin^4\beta + \cos^2\alpha_1\sin^2\beta + Z_1^2(S)/\ell_1^2]$$
$$/ [2\cos\alpha_1\sin^2\beta].$$

FIGURE 8. Definition of the angle α_i'.


β
A
ℓ_i
β
B
K
$\alpha_{i,i}$
β
F
B'
Z
Z

As shown in Fig.9, we have:

(108) $\cos\gamma_i = \cos[\alpha(AB) + \alpha_i']$.

So finally, we can obtain the other coordinates of the point B' as follows:

(109) $X_i(S) = x_i - |BK|\cos\gamma_i = x_i - \ell_i\cos\alpha_i\sin^2\beta\cos\gamma_i$,

(110) $Y_i(S) = y_i - |BK|\sin\gamma_i = y_i - \ell_i\cos\alpha_i\sin^2\beta\sin\gamma_i$.

From the above geometrical relationship we have obtain the coordinats at the point B'$_i$: X_i, Y_i, Z_i ($i = 2$ or 3). Also we may get $\Delta\beta$ and $\Delta\alpha$ based on Eq.(95) and (96) using the perturbed quantities. Therefore, we are now able to obtain the Gaussian coefficients as follows:

$$E(S) = [(X_2(S)-x_1)/\Delta\beta]^2 + [(Y_2(S)-y_1)/\Delta\beta]^2 + [(Z_2(S)-0)/\Delta\beta]^2,$$

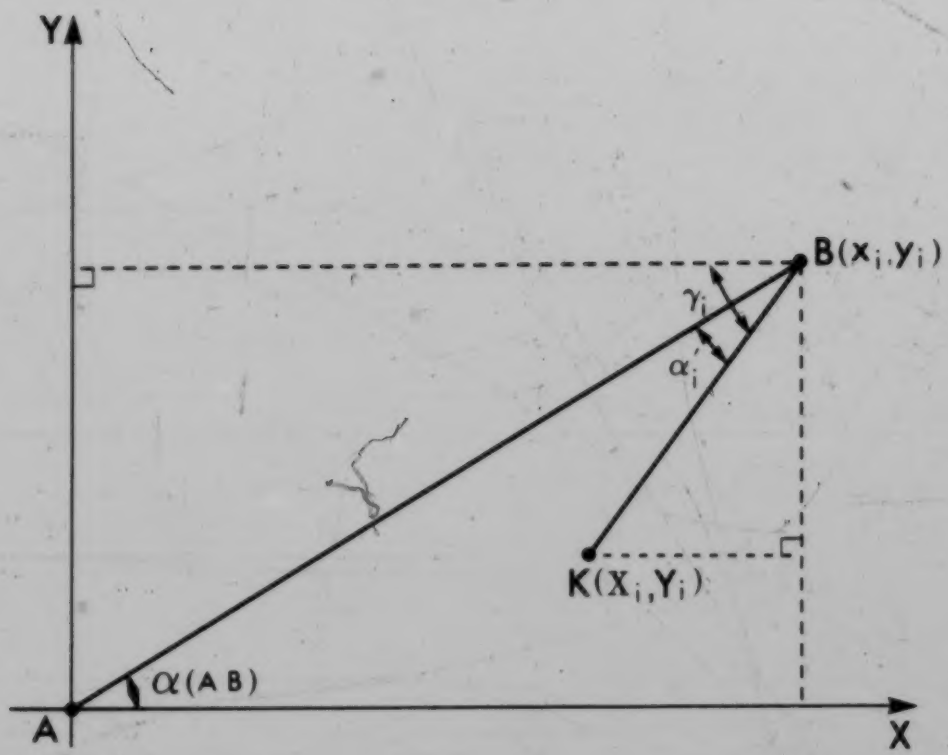
$$G(S) = [(X_3(S)-x_1)/\Delta\alpha]^2 + [(Y_3(S)-y_1)/\Delta\alpha]^2 + [(Z_3(S)-0)/\Delta\alpha]^2, \quad (111)$$

$$F(S) = [(X_2(S)-x_1)/\Delta\beta][(X_3(S)-x_1)/\Delta\alpha] + [(Y_2(S)-y_1)/\Delta\beta][(Y_3(S)-y_1)/\Delta\alpha] + [Z_2(S)/\Delta\beta][Z_3(S)/\Delta\alpha],$$

and the Jacobian, J, at the receiver point A:

(112) $J(S) = [E(S)G(S) - F(S)^2]^{1/2}$.

FIGURE 9. Definition of the angle γ_i and determination of the coordinates X_i and Y_i.


Y
B(x_i, y_i)
γ_i
α_i
K(X_i, Y_i)
α(A B)
A
X

At the taking off point (source), the coordinate, $X_i(s_0)$, $Y_i(s_0)$, $Z_i(s_0)$, where the perturbed ray intersects with the reference wavefront, could be obtained directly by solving Eq.(99) - (100). So we are able to obtain the Jacobian at the taking off point, $J(s_0)$, too.

Putting $J(S)$ and $J(s_0)$ into Eq.(42) we may determine the amplitude of the reflected ray at the receiver A when we know some of the following relative quantities:

$$\Phi(S) = R_1R_2\{J(s_0)\rho(s_0)V(s_0)/[J(S)\rho(S)V(S)]\}^{1/2}\Phi(s_0).$$

In the case of a medium with interfaces, we have to consider the boundary conditions at the interfaces concerning the ray path and the ray amplitude. V. Cerveny (1979) and I. Psencik (1972) have discussed them in detail. Their results are suitable for incorporation into our formulae.

10. Conclusion

From the above discussions we should obtain the following results:

(1) We may use ray path equation and the energy equation instead of the vectoral wave equation to study the kinematic and dynamic characteristics of the wave field in any weakly inhomogeneous medium, where the following restrictions are satisfied:

$$|L\ \nabla\rho/\rho| << 1,$$
$$|L\ \nabla\lambda/\lambda+2\mu| << 1,$$
$$|L\ \nabla\mu/\lambda+2\mu| << 1.$$

(2)The computer program ZH1 listed in Appendix I may be used to solve two point seismic ray tracing problem in a 2-D. medium and produce all kinematic characteristics of the wave field considered. The computation accuracy and efficiency of program ZH1 are highly reasonable.

(3) Based on the solution of two point ray tracing problem we described a method to compute the ray amplitudes in order to study dynamic characteristics of wave field. This would be an interesting project to follow up.

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solving two point seismic ray tracing problems in a heterogeneous and isotropic medium, U. S. Geol. Surv., Open-File Rept. 76-707, 66 pp.

APPENDIX I

```
/COMPILE SIZE=60P
BEGIN
INTEGER M, N;
STRING DATA;
REAL Z0, Z1, Z2, ALF1, ALF2, ALF3, ALF4, TOL;
READ(Z0, Z1, Z2, ALF1, ALF2, ALF3, ALF4);
M := 2;
N := 21;
TOL:= 1.0'-3;
BEGIN
REAL A, B, EPSMAC, EPS, ERROLD, ERRNEW, C1, HMAX,
     RABS, REOLD, SUM, TEM, AUXI, U1;
INTEGER KJH, JERROR, I, J, NUK, KMAX, N1, I1, IERROR, INWT,
        K1, NU2, K, NUINT, N02, N021, L, JK, AEMEK;
REAL ARRAY ABT, ALPHA(1::M);
REAL ARRAY A1, B1(1::M, 1::M);
REAL ARRAY AA, BB, CC(1::50);
LOGICAL DIVNEW;
PROCEDURE FF(REAL ARRAY X(*); REAL ARRAY Y(*,*);
                INTEGER VALUE N; REAL ARRAY F(*,*));
BEGIN
FOR I := 1 UNTIL N DO
BEGIN
F(1,I) := Y(2,I);
F(2,I) := (1+Y(2,I)**2) * (Z1*Y(2,I)-Z2)
          /(Z0+Z1*X(I)+Z2*Y(1,I));
END;
END FF;
PROCEDURE COEGE(INTEGER VALUE N, NP; REAL ARRAY C, BB(*);
                REAL VALUE E);
BEGIN
REAL ARRAY ALF(1::50);
INTEGER I, NN, LL, J, K, KM1, JM1;
REAL XKIN;
FOR I := 1 UNTIL N DO
C(I) := BB(I);
FOR I := 1 UNTIL N DO
ALF(I) := I - NP - E;
NN := N -1;
FOR I := 1 UNTIL NN DO
BEGIN
LL := N - I;
FOR J := 1 UNTIL LL DO
BEGIN
K := N - J + 1;
C(K) := C(K) - ALF(I) * C(K-1)
END;
END;
FOR I := 1 UNTIL NN DO
BEGIN
K := N - I;
XKIN := 1.0/K;
KM1 := K + 1;
FOR J := KM1 UNTIL N DO
BEGIN
C(J) := C(J) * XKIN;
JM1 := J - 1;
C(JM1) := C(JM1) - C(J);
END;
END;
```

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```
61      END COEGE;
62      PROCEDURE U2DCGS (INTEGER VALUE K, P, Q, N, M; REAL ARRAY
63                A(*); REAL ARRAY Y, S(*,*); REAL VALUE EE;
64                PROCEDURE COEGEN; INTEGER RESULT IERROR);
65      BEGIN
66      REAL ARRAY C(1::50);
67      INTEGER I, KK1, KK, KMID, KMID1, L, IT, NF, II, J, KMIDP1;
68      REAL ACUM;
69      IF (K > (N +1 -Q) DIV P) OR (P < 1) OR (K < 1) THEN GOTO S100;
70      IF (Q = 0) THEN GOTO S10;
71      FOR I := 1 UNTIL Q DO A(I) := 0;
72      S10: KK1 := Q + P * K;
73      KK := KK1 - 1;
74      KMID := KK1 DIV 2;
75      IERROR := 0;
76      KMID1 := KMID - 1;
77      IF (KMID < 1) THEN GOTO S25;
78      FOR I := 1 UNTIL KMID1 DO
79      BEGIN
80      COEGEN(KK1, I, C, A, EE);
81      FOR L := 1 UNTIL M DO
82      BEGIN
83      ACUM := 0;
84      FOR J := 1 UNTIL KK1 DO
85      ACUM := ACUM + C(J) * Y(L,J);
86      S(L,I) := ACUM;
87      END;
88      END;
89      S25: COEGEN(KK1, KMID, C, A, EE);
90      NF := N + 1 - KK1 + KMID;
91      FOR I := KMID UNTIL NF DO
92      BEGIN
93      II := I - KMID;
94      FOR L := 1 UNTIL M DO
95      BEGIN
96      ACUM := 0;
97      FOR J := 1 UNTIL KK1 DO
98      ACUM := ACUM + C(J) * Y(L, II+J);
99      S(L,I) := ACUM;
100     END;
101     END;
102     KMIDP1 := KMID + 1;
103     FOR I := KMIDP1 UNTIL KK DO
104     BEGIN
105     COEGEN(KK1, I, C, A, EE);
106     II := N - KK;
107     FOR L := 1 UNTIL M DO
108     BEGIN
109     ACUM := 0;
110     FOR J := 1 UNTIL KK1 DO
111     ACUM := ACUM + C(J) * Y(L, II + J);
112     S(L,I + II) := ACUM;
113     END;
114     END;
115     GOTO S20;
116     S100 : IERROR := 1;
117     S20:
118     END U2DCGS;
119     PROCEDURE JACOBB(REAL VALUE XX; REAL ARRAY YY(*);
120               REAL ARRAY XJAC(*,*));
```

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```
121     BEGIN
122     XJAC(1,1) := 0;
123     XJAC(1,2) := 1.0;
124     XJAC(2,1) := (1+YY(2)**2)*(Z2**2 - Z1*Z2*YY(2))
125                  /(Z0+Z1*XX + Z2*YY(1))**2;
126     XJAC(2,2) := (Z1-2*Z2*YY(2)+3*Z1*YY(2)**2)
127                  /(Z0+Z1*XX+Z2*YY(1));
128     END JACOBB;
129     PROCEDURE GS( INTEGER VALUE M,N; REAL ARRAY A(*,*));
130     BEGIN
131     INTEGER I0, J0, NM, I, J, K, TI;
132     REAL C, T;
133     INTEGER ARRAY Z(1::M);
134     NM := N + M;
135     FOR K := 1 UNTIL M DO Z(K) := K;
136     FOR K := 1 UNTIL M DO
137     BEGIN
138     C :=0;
139     FOR I := K UNTIL M DO
140     FOR J := K UNTIL M DO
141     IF ( ABS(A(I,J)) > ABS(C)) THEN
142     BEGIN
143     C := A(I,J);
144     I0 := I;
145     J0 := J
146     END;
147     IF ( ABS(C) < 1.0'-20) THEN
148     BEGIN
149     WRITE ( "SMOLLER" );
150     GOTO SEND;
151     END;
152     IF ( I0 = K ) THEN GOTO L1;
153     FOR J := 1 UNTIL NM DO
154     BEGIN
155     T := A(K,J);
156     A(K,J) := A(I0,J);
157     A(I0,J) := T
158     END;
159     L1: IF (J0 = K ) THEN GOTO L2;
160     FOR I := K UNTIL M DO
161     BEGIN
162     T := A(I,K);
163     A(I,K) := A(I,J0);
164     A(I,J0) := T
165     END;
166     TI := Z(K);
167     Z(K) := Z(J0);
168     Z(J0) := TI;
169     L2: C := 1/C;
170     FOR J := K+1 UNTIL NM DO
171     A(K,J) := A(K,J) * C;
172     FOR I := K+1 UNTIL M DO
173     FOR J := K+1 UNTIL NM DO
174     A(I,J) := A(I,J) - A(I,K) * A(K,J)
175     END;
176     FOR I := M + 1 UNTIL NM DO
177     FOR K := M - 1 STEP -1 UNTIL 1 DO
178     BEGIN
179     C := A(K,I);
180     FOR J := K + 1  UNTIL M DO
```

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```
181     C := C - A(K,J) * A(J,I);
182     A(K,I) := C
183     END;
184     FOR K:= 1 UNTIL M DO
185     BEGIN
186     FOR I := K + 1 UNTIL M DO
187     IF ( Z(I) = K ) THEN
188     BEGIN
189     FOR J := M + 1 UNTIL NM DO
190     BEGIN
191     T := A(K,J);
192     A(K,J) := A(I,J);
193     A(I,J) := T;
194     END;
195     Z(I) := Z(K);
196     GOTO L3
197     END;
198     L3:
199     END
200     END GS;
201     PROCEDURE SYSLIN( INTEGER VALUE M,N; REAL ARRAY X(*);
202               REAL ARRAY Y(*,*); REAL ARRAY H(*); REAL ARRAY
203               RES, A1, B1(*,*); PROCEDURE JACOB, GS2; REAL
204               ARRAY UU(*,*));
205     BEGIN
206     INTEGER I, J, M2, MPN, M1, K1, K, L;
207     REAL X1, XL, SUM;
208     REAL ARRAY H2(1::N-1);
209     REAL ARRAY Y1, U(1::M);
210     REAL ARRAY JAB, S, R(1::M, 1::M);
211     REAL ARRAY T, VM, AUX, AB1(1::M,1::M+1);
212     REAL ARRAY A(1::M,1::2 * M + 1);
213     REAL ARRAY V(1::N * M, 1::M + 1);
214     REAL ARRAY U1(1::M * N);
215     X1 := X(1);
216     FOR I := 1 UNTIL M DO
217     Y1(I) := Y(I,1);
218     JACOB(X1, Y1, JAB);
219     M2 := M * M;
220     MPN := M * N;
221     M1 := M + 1;
222     FOR I := 1 UNTIL M DO
223     T(I,M1) := RES(I,2);
224     FOR L := 2 UNTIL N DO
225     BEGIN
226     FOR J := 1 UNTIL M DO
227     U(J) := Y(J, L);
228     H2(L - 1) := 0.5 * H(L - 1);
229     FOR J := 1 UNTIL M DO
230     FOR I := 1 UNTIL M DO
231     BEGIN
232     S(I,J) := H2(L - 1) * JAB(I,J);
233     IF (I=J) THEN S(I,J) := S(I,J) + 1.0;
234     END;
235     XL := X(L);
236     JACOB(XL, U, JAB);
237     FOR I := 1 UNTIL M DO
238     FOR J := 1 UNTIL M DO
239     BEGIN
240     R(I,J) := -H2(L-1) * JAB(I,J);
```

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```
241     IF (I=J) THEN R(I,J) := R(I,J) + 1.0;
242     END;
243     IF (L > 2) OR (L < 2) THEN GOTO S1300;
244     FOR I := 1 UNTIL M DO
245     FOR J := 1 UNTIL M DO
246     T(I,J) := - S(I,J);
247     GOTO S1700;
248     S1300 :
249     FOR K1 := 1 UNTIL M1 DO
250     FOR I := 1 UNTIL M DO
251     BEGIN
252     SUM := 0;
253     FOR J := 1 UNTIL M DO
254     SUM := SUM + S(I,J) * VM(J,K1);
255     T(I,K1) := SUM;
256     END;
257     FOR I := 1 UNTIL M DO
258     T(I,M1) := T(I,M1) + RES(I,L);
259     S1700:
260     FOR I := 1 UNTIL M DO
261     FOR J := 1 UNTIL M DO
262     A(I,J) := R(I,J);
263     FOR J := 1 UNTIL M1 DO
264     FOR I := 1 UNTIL M DO
265     A(I,M + J) := T(I,J);
266     GS2(M, M1, A);
267     FOR J := 1 UNTIL M1 DO
268     FOR I := 1 UNTIL M DO
269     VM(I,J) := A(I,M+J);
270     FOR J := 1 UNTIL M1 DO
271     FOR I := 1 UNTIL M DO
272     V((L-1) * M + I, J) := VM(I,J);
273     END;
274     FOR J := 1 UNTIL M1 DO
275     FOR I := 1 UNTIL M DO
276     BEGIN
277     SUM := 0;
278     FOR K := 1 UNTIL M DO
279     SUM := SUM + B1(I,K) * VM(K,J);
280     AUX(I,J) := SUM;
281     END;
282     FOR J := 1 UNTIL M DO
283     FOR I := 1 UNTIL M DO
284     R(I,J) := A1(I,J) - AUX(I,J);
285     FOR I := 1 UNTIL M DO
286     U1(I) := RES(I,1) - AUX(I,M1);
287     FOR I := 1 UNTIL M DO
288     FOR J := 1 UNTIL M DO
289     AB1(I,J) := R(I,J);
290     FOR I := 1 UNTIL M DO
291     AB1(I,M + 1) := U1(I);
292     GS2(M, 1, AB1);
293     FOR I := 1 UNTIL M DO
294     U1(I) := AB1(I, M + 1);
295     FOR I := M1 UNTIL MPN DO
296     BEGIN
297     SUM := V(I,M1);
298     FOR J := 1 UNTIL M DO
299     SUM := SUM - V(I,J) * U1(J);
300     U1(I) := SUM;
```

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```
301     END;
302     FOR J := 1 UNTIL N DO
303     BEGIN
304     K1 := (J - 1) * M;
305     FOR I := 1 UNTIL M DO
306     UU(I,J) := U1(K1 + I);
307     END
308     END;
309     NEWLINE(13);
310     WRITE("                    VELOCITY MODEL:");
311     NEWLINE(2);
312     WRITE("                    V=Z0+Z1*X+Z2*Z");
313     IOCONTROL("SPACE");
314     WRITE("                    Z0=",Z0);
315     WRITEON("Z1=",Z1);
316     WRITEON("Z2=",Z2);
317     NEWLINE(2);
318     WRITE("                    X(0)=",ALF1);
319     WRITEON("Z(0)=",ALF2);
320     WRITE("                    X(1)=",ALF3);
321     WRITEON("Z(1)=",ALF4);
322     IOCONTROL("EJECT");
323     A := ALF1;
324     B := ALF3;
325     EPSMAC := 1.0'-40;
326     JERROR := 0 ;
327     DIVNEW := FALSE ;
328     KJH := 0;
329     NUK := 0;
330     FOR I := 1 UNTIL M DO
331     BEGIN
332     ALPHA(I) := 0;
333     FOR J := 1 UNTIL M DO
334     BEGIN
335     A1(I,J) := 0;
336     B1(I,J) := 0;
337     END;
338     END;
339     A1(1,1) := 1.0;
340     B1(2,1) := 1.0;
341     ALPHA(1) := ALF2;
342     ALPHA(2) := ALF4;
343     AEMEK := 0;
344     JK := -10;
345       LLL3:
346     JK := JK + 10;
347     BEGIN
348     REAL ARRAY H(1::N-1);
349     REAL ARRAY X(1::N);
350     REAL ARRAY Y, RES(1::M, 1::N);
351     INTEGER ARRAY IQJ(1::N-1);
352     PROCEDURE ENDPU;
353     BEGIN
354     REAL ARRAY SSUM,TSUM(1::N);
355     SSUM(1) := 0.0;
356     TSUM(1) := 0.0;
357     FOR I := 2 UNTIL N DO
358     BEGIN
359     U1 := SQRT((X(I)-X(I-1))**2+(Y(1,I)-Y(1,I-1))**2);
360     SSUM(I) := SSUM(I-1)+U1;
```

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```
361     TSUM(I) := TSUM(I-1)+U1/(Z0+0.5*Z1*(X(I)+X(I-1))+
362             0.5*Z2*(Y(1,I)+Y(1,I-1)));
363     END;
364     NEWLINE(3);
365     R_W := 10;
366     I_W := 3;
367     WRITE("                N=", N);
368     WRITE("                TOL=", TOL, "(", ERRNEW, ")");
369     NEWLINE(1);
370     S_W := 5;
371     R_D := 3;
372     R_FORMAT := "F";
373     K1 := (N-1) DIV 20;
374     FOR J := 1 STEP K1 UNTIL N DO
375     BEGIN
376     WRITE("                N=", J);
377     WRITEON("X=", X(J));
378     WRITEON("Z=", Y(1,J));
379     SUM := ARCTAN(Y(2,J))*180/3.14159265;
380     WRITEON("Q=",SUM);
381     WRITEON("T=",TSUM(J));
382     WRITEON("S=",SSUM(J));
383     END;
384     K1 := (N-1) REM K1;
385     IF (K1=0) THEN GOTO SENDPU;
386     WRITE("                N=", N);
387     WRITEON("X=",X(N));
388     WRITEON("Z=",Y(1,N));
389     SUM := ARCTAN(Y(2,N))*180/3.14159265;
390     WRITEON("Q=",SUM);
391     WRITEON("T=",TSUM(N));
392     WRITEON("S=",SSUM(N));
393     SENDPU:
394     END;
395     K := 0;
396     BEGIN
397     REAL ARRAY F, UU(1::M, 1::N);
398     REAL ARRAY SK(1::M, 1::N-1);
399     REAL HH;
400     IF DIVNEW THEN GOTO S40;
401     IF (KJH >= 1) THEN GOTO S120;
402       S40:
403     TEM := (ALF2 - ALF4)/(ALF3 - ALF1);
404     SUM := (ALF3 - ALF1)/(N-1);
405     FOR J := 1 UNTIL N DO
406     BEGIN
407     Y(1,J) := ALF2 - TEM * SUM*(J-1);
408     Y(2,J) := -TEM;
409     END;
410     BB(1) := 1.0;
411     FOR I := 2 UNTIL 50 DO BB(I) := 0;
412     HH := (B-A)/(N-1);
413     IF (EPSMAC > 0.01 * HH ** 4) THEN EPS := EPSMAC
414     ELSE EPS := 0.01 * HH ** 4;
415     NUK := 0;
416     X(1) := A;
417     X(N) := B;
418     FOR I := 2 UNTIL N-1 DO
419     X(I) := A + (I-1) * HH;
420     FOR I := 1 UNTIL N-1 DO
```

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```
421     H(I) := HH;
422       S120:
423     ERROLD := 1.0;
424     C1 := 0.8;
425     FOR I := 1 UNTIL M DO
426     FOR J := 1 UNTIL N-1 DO
427     SK(I,J) := 0;
428     IF (KJH >= 1) THEN
429     BEGIN
430     REWIND(7);
431     FOR J := 1 UNTIL N DO
432     FOR I := 1 UNTIL M DO
433     GET(7,NULL,Y(I,J));
434     FOR I := 1 UNTIL N-1 DO
435     GET(7,NULL,H(I));
436     RELEASE(7);
437     X(1) := A;
438     X(N) := B;
439     FOR I := 2 UNTIL N-1 DO
440     X(I) := X(I-1) + H(I-1);
441     END;
442     KMAX := (N-2) DIV 2;
443     N1 := N-1;
444     HMAX := 1/N1;
445     IF (NUK = 0) THEN GOTO S405;
446     FF(X, Y, N, F);
447     U2DCGS(NUK, 2, 2, N1, M, AA, F, RES, 0.5, COEGE, IERROR);
448     FOR J := 1 UNTIL N1 DO
449     FOR I := 1 UNTIL M DO
450     SK(I,J) := H(J) * RES(I,J);
451       S405:
452     INWT := 0;
453     REOLD := 1.0;
454     DIVNEW := FALSE;
455     IF (KMAX > NUK + 1) THEN GOTO S410;
456     NUK := NUK + 1;
457     GOTO SPAS;
458       S410:
459     RABS := 0;
460     FOR I := 1 UNTIL M DO
461     BEGIN
462     SUM := ALPHA(I);
463     FOR J := 1 UNTIL M DO
464     SUM := SUM - A1(I,J) * Y(J,1) - B1(I,J) * Y(J,N);
465     RES(I,1) := SUM;
466     TEM := ABS(RES(I,1));
467     IF (TEM > RABS) THEN RABS := TEM;
468     END;
469     FF(X, Y, N, F);
470     FOR I := 2 UNTIL N DO
471     FOR J := 1 UNTIL M DO
472     BEGIN
473     RES(J,I) := -Y(J,I) + Y(J,I-1) + H(I-1)/2 * (F(J,I)
474                 + F(J,I-1)) + SK(J,I-1);
475     TEM := ABS(RES(J,I));
476     IF (TEM > RABS) THEN RABS := TEM;
477     END;
478     IF (RABS>1.0) AND (INWT=1) THEN
479     BEGIN
480     DIVNEW := TRUE;
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481     N := N+10;
482     GOTO LLL3;
483     END;
484     IF (RABS>REOLD) AND (INWT>1) THEN GOTO S1500;
485     IF (INWT>=3) THEN GOTO S1500;
486     SYSLIN(M, N, X, Y, H, RES, A1, B1, JACOBB, GS, UU);
487     REOLD := RABS;
488     FOR I := 1 UNTIL M DO
489     FOR J := 1 UNTIL N DO
490     Y(I,J) := Y(I,J) + UU(I,J);
491     INWT := INWT + 1;
492     GOTO S410;
493       S1500:
494     NUK := NUK + 1;
495     NU2 := 2 * NUK + 1;
496     AA(NU2) := -NUK / (2 ** (2 * NUK - 1) * NU2);
497     AA(NU2 + 1) := 0;
498     U2DCGS(NUK, 2, 2, N1, M, AA, F, RES, 0.5, COEGE, IERROR);
499     IF (IERROR = 1) THEN GOTO SPAS;
500     FOR J := 1 UNTIL N1 DO
501     FOR I := 1 UNTIL M DO
502     BEGIN
503     AUXI := RES(I,J) * H(J);
504     RES(I,J) := SK(I,J) - AUXI;
505     SK(I,J) := AUXI;
506     END;
507     SYSLIN(M, N, X, Y, H, RES, A1, B1, JACOBB, GS, UU);
508     ERRNEW := 0;
509     FOR I := 1 UNTIL M DO
510     ABT(I) := 0.0;
511     FOR J := 1 UNTIL N DO
512     FOR I := 1 UNTIL M DO
513     BEGIN
514     U1 := ABS(UU(I,J));
515     IF (U1 > ABT(I)) THEN ABT(I) := U1;
516     END;
517     FOR J := 1 UNTIL M DO
518     BEGIN
519     IF (ABT(J) > ERRNEW) THEN ERRNEW := ABT(J);
520     END;
521     IF (ERRNEW <= TOL) THEN
522     BEGIN
523     ENDPU;
524     GOTO SEND;
525     END;
526     CC(NUK) := ERRNEW;
527     IF (ERRNEW <= 0.1 * ERROLD) THEN GOTO S2550;
528     IF (ERRNEW > C1 * ERROLD) THEN GOTO SPAS;
529     C1 := 0.5 * C1;
530       S2550:
531     ERROLD := ERRNEW;
532     IF (EPS = EPSMAC) THEN
533     BEGIN
534     WRITE("ERROLD=", ERROLD);
535     WRITE("NEED CHANGE TOL");
536     STOP(NULL);
537     END;
538     IF (EPSMAC > 0.01 * HMAX ** 2 * ERROLD) THEN
539     EPS := EPSMAC ELSE EPS := 0.01 * HMAX ** 2 *ERROLD;
540     GOTO S405;
```

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541       SPAS:
542     IF (JK = 140) THEN
543     BEGIN
544     WRITE("ERROLD=", ERROLD);
545     WRITE("NEED CHANGE TOL");
546     GOTO SEND;
547     END;
548     K := NUK - 1;
549     I1 := 50;
550     GOTO AEME;
551     END;
552       AEME:
553     BEGIN
554     REAL ARRAY EJ(1::N-1);
555     INTEGER IQ,NMAX,IQSR;
556     REAL SAKM, BMA, ELEV,LMTA,ESR,ALG;
557       SIQ:
558     TEM := TOL;
559     SAKM := 1/(2*K+2);
560     ELEV := TEM ** ROUND(SAKM);
561     U1 := 0;
562     FOR J := 1 UNTIL N1 DO
563     BEGIN
564     AUXI := H(J) ** (2*K+2);
565     SUM := 0;
566     FOR I := 1 UNTIL M DO
567     BEGIN
568     TEM := ABS(RES(I,J));
569     IF (TEM > SUM) THEN SUM := TEM;
570     END;
571     SUM := SUM/AUXI;
572     IF (SUM > U1) THEN U1 :=SUM;
573     END;
574     LMTA := U1/I1 ** (2*K+2);
575     FOR J := 1 UNTIL N1 DO
576     BEGIN
577     AUXI := H(J) **(2*K+2);
578     SUM := 0;
579     FOR I := 1 UNTIL M DO
580     BEGIN
581     TEM := ABS(RES(I,J));
582     IF (TEM > SUM) THEN SUM := TEM;
583     END;
584     SUM := SUM / AUXI;
585     IF (SUM >= LMTA) THEN U1 := SUM ** ROUND(SAKM);
586     IF (LMTA > SUM) THEN U1 := LMTA ** ROUND(SAKM);
587     EJ(J) := H(J) * U1 ;
588     END;
589     FOR J := 1 UNTIL N1 DO
590     IQJ(J) := ROUND(EJ(J) / ELEV);
591     IQ := 0;
592     FOR J := 1 UNTIL N1 DO
593     BEGIN
594     IF (IQJ(J) = 0) THEN IQ :=IQ + IQJ(J) ELSE
595     IQ := IQ + IQJ(J) - 1;
596     END;
597     NMAX := 2000 DIV M;
598     U1 := NMAX;
599     ALG := 1.4;
600     RABS := NMAX - N;
```

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```
601     IF (RABS > 70.0) THEN RABS := 70.0;
602       SJU:
603     IF (IQ < 0.04 * N) THEN GOTO SLS;
604     IF (IQ > RABS) THEN GOTO SGR;
605     IF (IQ >= 0.04 * N) AND (IQ <= RABS) THEN
606     BEGIN
607     AEMEK := K;
608     GOTO S2600;
609     END;
610       SLS:
611     SUM := 1.4 * N;
612     IF (SUM < U1) THEN TEM := SUM ELSE TEM := U1;
613     SUM := TEM - N - 1;
614     IQSR := ROUND(SUM);
615     ESR := (N+IQ+1) / (N+IQSR+1) * ELEV;
616     FOR J := 1 UNTIL N1 DO
617     IQJ(J) := ROUND(EJ(J) / ESR);
618     IQ := 0;
619     FOR J := 1 UNTIL N1 DO
620     BEGIN
621     IF (IQJ(J) = 0) THEN IQ := IQ+IQJ(J) ELSE
622     IQ := IQ + IQJ(J) - 1;
623     END;
624     IF (IQ < 0.04 * N) THEN
625     BEGIN
626     IF (K > 0) THEN K := K-1;
627     IF (K = 0) THEN
628     BEGIN
629     IF ((2*N - 1) <= NMAX) THEN GOTO S026 ELSE
630     BEGIN
631     WRITE("NMAX SO SMALL THAT STOP");
632     GOTO SEND;
633     END;
634     END;
635     IF (K > 0) THEN GOTO SIQ;
636     END;
637     IF (IQ >= 0.04 * N) THEN GOTO SJU;
638       SGR:
639     SUM := ALG * N;
640     IF (SUM < U1) THEN TEM := SUM ELSE
641     TEM := U1;
642     SUM := TEM - N - 1;
643     IQSR := ROUND(SUM);
644     ESR := (N+IQ+1) / (N+IQSR+1) * ELEV;
645     FOR J := 1 UNTIL N1 DO
646     IQJ(J) := ROUND((EJ(J) / ESR));
647     IQ := 0;
648     FOR J := 1 UNTIL N1 DO
649     BEGIN
650     IF ( IQJ(J) = 0) THEN IQ := IQ + IQJ(J) ELSE
651     IQ := IQ + IQJ(J) -1;
652     END;
653     IF (IQ > RABS) THEN
654     BEGIN
655     ALG := ALG - 0.1;
656     IF (ALG >= 1.1) THEN GOTO SGR;
657     END;
658     IF (IQ <= RABS) THEN GOTO SJU;
659     END;
660       S2600:
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661     BEGIN
662     REAL ARRAY SK(1::M,1::N-1);
663     I1 := 1;
664     FOR I := 1 UNTIL N1 DO
665     BEGIN
666     IF (IQJ(I) > 0) THEN
667     I1 := I1 + 1 + IQJ(I) -1 ELSE
668     I1 := I1 +1;
669     END;
670     BEGIN
671     REAL ARRAY Y1(1::M,1::I1);
672     REAL ARRAY H1(1::I1-1);
673     REAL ARRAY CC1(1::50);
674     REAL HH;
675     INTEGER J1;
676     KJH := KJH +1;
677     NUINT := NUK;
678     FOR I := 1 UNTIL M DO
679     FOR J := 1 UNTIL N DO
680     RES(I,J) := Y(I,J);
681     N1 := N - 1;
682     L := 1;
683     FOR J := 1 UNTIL N1 DO
684     BEGIN
685     IF (IQJ(J) > 1) THEN
686     BEGIN
687     K1 := L;
688     L := L + IQJ(J) - 1;
689     FOR I := 1 UNTIL M DO Y1(I,K1) := RES(I,J);
690     FOR J1 := K1+1 UNTIL L DO
691     BEGIN
692     U1 := 1/IQJ(J) * (J1-K1);
693     U2DCGS(NUINT,2,0,N1,M,BB,Y,SK,U1,COEGE,IERROR);
694     FOR I := 1 UNTIL M DO
695     Y1(I,J1) := SK(I,J);
696     END;
697     END;
698     IF (IQJ(J) <= 1) THEN
699     BEGIN
700     FOR I := 1 UNTIL M DO
701     Y1(I,L) := RES(I,J);
702     END;
703     L := L+1;
704     END;
705     FOR I := 1 UNTIL M DO
706     Y1(I,I1) := RES(I,N);
707     L := 1;
708     FOR I := 1 UNTIL N1 DO
709     BEGIN
710     IF (IQJ(I) > 1) THEN
711     BEGIN
712     HH := H(I) / IQJ(I);
713     K1 := L;
714     L := L+IQJ(I)-1;
715     FOR J := K1 UNTIL L DO
716     H1(J) := HH;
717     END;
718     IF (IQJ(I) <= 1) THEN
719     H1(L) := H(I);
720     L := L +1;
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721     END;
722     ASSIGN(7,"DATA");
723     FOR J := 1 UNTIL I1 DO
724     FOR I := 1 UNTIL M DO
725     PUT(7, NULL, Y1(I,J));
726     FOR I := 1 UNTIL (I1 - 1) DO
727     PUT(7, NULL, H1(I));
728     PUT(7, NULL, 0.0);
729     FOR I := 1 UNTIL NUK DO
730     CC1(I) := CC(I);
731     K1 := 0;
732     FOR I := 1 UNTIL N1 DO
733     IF (IQJ(I) > K1) THEN K1 := IQJ(I);
734     NUK := 0;
735     IF (EPSMAC > 0.01 * (1/(I1-1)) ** 4) THEN
736     EPS := EPSMAC ELSE EPS := 0.01 * (1/(I1-1)) ** 4;
737     N := I1;
738     GOTO LLL3;
739     END;
740     END;
741       S026:
742     BEGIN
743     REAL ARRAY Y1(1::M,1::2*N-1);
744     REAL ARRAY SK(1::M,1::N);
745     REAL ARRAY H1(1::2*(N-1));
746     REAL ARRAY CC1(1::50);
747     KJH := KJH + 1;
748     HMAX := H(N1);
749     FOR I := 1 UNTIL N-1 DO
750     BEGIN
751     TEM := ABS(H(I));
752     IF (TEM > HMAX) THEN HMAX := TEM;
753     END;
754     NUINT := NUK;
755     FOR I := 1 UNTIL NUK DO
756     CC1(I) := CC(I);
757     K := 0;
758     FOR I := 1 UNTIL NUK DO
759     BEGIN
760     K := K-1;
761     CC1(I) := CC1(I) * 4.0 ** K;
762     END;
763     IF (ERRNEW < ERROLD) THEN ERROLD := ERRNEW;
764     K := 0;
765     FOR I := 1 UNTIL NUK DO
766     BEGIN
767     K := K+1;
768     IF (ERROLD < CC1(I)) THEN GOTO S270;
769     GOTO S275;
770       S270:
771     END;
772       S275:
773     NUK := K -1;
774     IF (EPSMAC > 0.005*HMAX**2*ERROLD) THEN EPS := EPSMAC
775     ELSE EPS := 0.005 * HMAX ** 2 * ERROLD;
776     FOR I := 1 UNTIL M DO
777     FOR J := 1 UNTIL N DO
778     RES(I,J) := Y(I,J);
779     N := 2 * N -1;
780     N02 := (N+1) DIV 2;
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```
781       N021 := N02 - 1;
782       U2DCGS(NUINT,2,0,N021,M,BB,Y,SK,0.5,COEGE,IERROR);     98
783       FOR J := 1 UNTIL N021 DO
784       FOR I := 1 UNTIL M DO
785       BEGIN
786       Y1(I,2*J-1) := RES(I,J);
787       Y1(I,2*J) := SK(I,J);
788       END;
789       FOR L := 1 UNTIL M DO
790       Y1(L,N) := RES(L,N02);
791       FOR I := 1 UNTIL N021 DO
792       BEGIN
793       H1(2*I-1) := H(I)/2;
794       H1(2*I) := H(I)/2;
795       END;
796       ASSIGN(7,"DATA");
797       FOR J := 1 UNTIL N DO
798       FOR I := 1 UNTIL M DO
799       PUT(7,NULL,Y1(I,J));
800       FOR I := 1 UNTIL N-1 DO
801       PUT(7,NULL,H1(I));
802       PUT(7,NULL,0.0);
803       GOTO LLL3;
804       END;
805       END;
806         SEND:
807       END;
808       END.
End of file
```

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